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A Uniqueness Theorem for Linear Wave Equations

WHITMAN Phillip[1] and YU Pin[2,*]

[1] *Department of Mathematics, Princeton University, Princeton 08540, New Jersey, USA.*
[2] *Mathematical Sciences Center, Tsinghua University, Haidian District, Beijing 100084, China.*

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Abstract. The classical Huygens' principle asserts that the initial data of a wave equation determines the wave propagation in the domain of dependence of the support of the data. We provide a converse version of this theorem.

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1 Introduction

1.1 Main results

We study uniqueness properties of the free wave equation

$$\Box\varphi=0, \tag{1.1}$$

on $\mathbb{R}^{n+1}$ where we assume φ is a smooth[†] solution. In view of the Huygens principle or standard energy estimates, if initial data vanishes[‡] on a set in the hyperplane $\{t=0\}$, the solution of (1.1) vanishes on the domain of dependence of this set. The purpose of the paper is to provide a converse version of this statement.

Let $r=\sqrt{x_1^2+\cdots+x_n^2}$ be the standard radius function. We use $B_{t_0}(r)\subset\{t=t_0\}$ to denote the n-dimensional ball of radius r centered at the origin of hyperplane $\{t=t_0\}$.

*Corresponding author. *Email addresses:* pwhitman@math.princeton.edu (P. Whitman), pin@math.tsinghua.edu.cn (P. Yu)

† From the proof of the main results, we can see that C^2 regularity is enough.
‡ Vanishing data means that both φ and $\partial_t\varphi$ vanish.

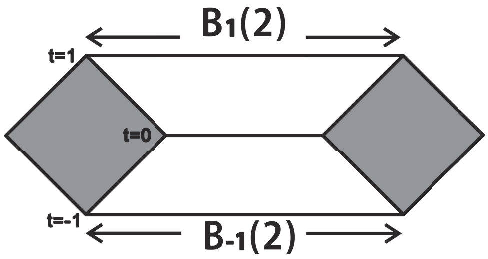

Figure 1:

The Huygens principle implies, if the data $(\varphi,\partial_t\varphi)|_{t=0}=0$ on the ball $B_0(3)$, then $(\varphi,\partial_t\varphi)$ vanishes on $B_1(2)$ and $B_{-1}(2)$. In converse, if one knows $(\varphi,\partial_t\varphi)$ vanishes on $B_1(2)$ and $B_{-1}(2)$, by using the Huygens principle, one can only say that at time slice $t=0$, $(\varphi,\partial_t\varphi)$ must vanish on $B_0(1)$. One may ask if it is possible to show more, say to determine the maximal domain on which the wave vanishes. Intuitively, this maximal domain in $\{t=0\}$ should be $B_0(3)$. The answer is yes: the knowledge of free waves on $B_1(2)$ and $B_{-1}(2)$ are enough to determine itself on $B_0(3)$. This is also pictured in Fig. 1 where we would like to show that the solution is also zero in the shaded region.

Proposition 1.1. *Assume that φ is a smooth solution of*

$$\Box\varphi=0.$$

If $(\varphi,\partial_t\varphi)$ vanishes on $B_{-1}(2)$ and $B_1(2)$, then it must vanish on $B_0(3)$.

If we study waves on $\mathbb{R}^{1+1}$, the above theorem is almost obvious: we can decompose the wave into outgoing and incoming components and show that each component is determined by its value on $B_1(2)$ and $B_{-1}(2)$. This proof also motivates the theorem on higher dimensions, but the proof is the current work. In higher dimensions, although we can not separate variable as in one dimension, we still have an explicit formulas, namely, we can obtain solutions via the method of spherical means. In odd dimensions, this connects the uniqueness problems directly to integrals on spheres (so called the spherical Radon transforms), especially the Helgason's support theorem. In particular, a similar (but different) result has been proved on an inverse Huygens principle by Helgason in [1]. In this work, he proved a support theorem for geodesic spheres is applied to prove an inverse Huygens principle for the normalized wave equation on hyperbolic space. The proof of the Proposition 1.1 in the current work does not depend on any explicit transformation formula or fundamental solutions and it can be easily generalized to other space-times, e.g. waves equations on Schwarzschild space-time.

In fact, we can state a slightly more general version of the theorem as follows:

Proposition 1.2. *Let $t_1<t_2$ be two distinct times and $r_1,r_2>0$ be two radii. Let $C_1\subset\mathbb{R}^{n+1}$ be the causal future of $t_1\times B(0,r_1)$, and $C_2\subset\mathbb{R}^{n+1}$ be the causal past of $t_2\times B(0,r_2)$. Then, if $r_1+r_2>t_2-t_1$ and φ solves the linear wave equation on $\mathbb{R}^{n+1}$ and vanishes on $t_1\times B(0,r_1)\cup t_2\times B(0,r_2)$, then φ vanishes on $C_1\cap C_2$.*