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FIXED-POINT FAST SWEEPING WENO METHODS FOR STEADY STATE SOLUTION OF SCALAR HYPERBOLIC CONSERVATION LAWS

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Abstract. Fast sweeping methods were developed in the literature to efficiently solve static Hamilton-Jacobi equations. This class of methods utilize the Gauss-Seidel iterations and alternating sweeping strategy to achieve fast convergence rate. They take advantage of the properties of hyperbolic partial differential equations (PDEs) and try to cover a family of characteristics of the corresponding Hamilton-Jacobi equation in a certain direction simultaneously in each sweeping order. In [16], the Gauss-Seidel idea and alternating sweeping strategy were adopted to the time-marching type fixed-point iterations to solve the static Hamilton-Jacobi equations, and numerical examples verified at least a 2 times acceleration of convergence even on relatively coarse grids. In this paper, we apply the same approach to solve steady state solution of hyperbolic conservation laws. We use numerical examples to verify that a 2 times acceleration of convergence is achieved. And the computational cost is exactly the same as the time-marching scheme at each iteration. Based on the Gauss-Seidel iterations, we explore the successive overrelaxation (SOR) approach to further improve the performance of our fixed-point sweeping methods.

Key words. fast sweeping methods, WENO methods, Jacobi iteration, Gauss-Seidel iteration, hyperbolic conservation laws, steady state.

1. Introduction

Steady state problems for hyperbolic conservation laws and related Hamilton-Jacobi (HJ) equations are common mathematical models appearing in many applications, such as fluid mechanics, optimal control, differential games, image processing and computer vision, geometric optics, etc. For these boundary value problems, their solution information propagates along characteristics starting from the boundary. A class of iterative methods, called fast sweeping (FS) methods [1, 6, 9, 12, 14, 16, 17, 18], take advantage of this property and try to cover a family of characteristics of the HJ equations in a certain direction simultaneously in each iteration. This iterative technique can achieve very fast convergence for computations of steady state solutions. Fast sweeping methods actually provide a general methodology / technique to accelerate the convergence of numerical schemes for steady state problems of hyperbolic type PDEs, although currently they are mostly used for solving HJ equations.

Since fast sweeping technique mainly takes advantage of the characteristics properties of hyperbolic PDEs to accelerate the iteration convergence, it is natural to apply this technique for solving steady states of general hyperbolic PDEs [2, 8]. In [16], we proposed fixed-point fast sweeping methods for static Hamilton-Jacobi equations. The fixed-point fast sweeping approach is based on the time marching approach and it has the advantages that the method is explicit and free of solving nonlinear equations, and it is straightforward to apply high order approximations and different numerical Hamiltonian for the general Hamilton-Jacobi equations.

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In this paper we apply this "explicit fast sweeping technique", or the "fixed-point fast sweeping method" to solve the steady state problems of hyperbolic conservation laws. The Gauss-Seidel idea and alternating sweeping strategy are adopted to the time marching approach to accelerate its convergence to steady states *without* any additional computational cost. Via numerical experiments, we verify this acceleration. In this paper, we use the standard third order finite difference weighted essentially non-oscillatory (WENO) scheme with Lax-Friedrichs flux splitting [5] as the representation of high order schemes for hyperbolic conservation laws. But this general approach can be directly applied to other schemes such as the residual distribution WENO schemes [3] or Runge-Kutta discontinuous Galerkin methods [4]. It can also be applied to other numerical fluxes such as Godunov flux, etc [10]. Based on the Gauss-Seidel iterations, we explore the successive overrelaxation (SOR) approach to further improve the performance of our fixed-point sweeping methods.

In Section 2, we describe the fixed-point fast sweeping WENO methods for solving hyperbolic conservation laws, based on Gauss-Seidel iterations and SOR iterations respectively. In Section 3, Numerical studies are performed to verify the faster convergence speed than the usual time-marching approach. Concluding remarks are given in Section 4.

2. Fixed-point fast sweeping WENO methods

Consider two-dimensional steady state problems of hyperbolic conservation laws with appropriate boundary conditions

$$f(u)_x + g(u)_y = h(u, x, y), \tag{1}$$

where u is the unknown function, f and g are flux functions, and h is the source term. A high order spatial discretization of (1) leads to a nonlinear system. In this paper, we use the third order finite difference WENO scheme with Lax-Friedrichs flux splitting [10] for the spatial discretization.

2.1. WENO discretization. For the hyperbolic terms $f(u)_x + g(u)_y$, the conservative finite difference scheme we use approximates the point values at a uniform (or smoothly varying) grid (x_i, y_j) in a conservative fashion. Namely, the derivative $f(u)_x$ at (x_i, y_j) is approximated along the line $y = y_j$ by a conservative flux difference

$$f(u)_x|_{x=x_i} \approx \frac{1}{\Delta x}(\hat{f}_{i+1/2} - \hat{f}_{i-1/2}), \tag{2}$$

where for the third order WENO scheme the numerical flux $\hat{f}_{i+1/2}$ depends on the three-point values $f(u_l)$, $l = i-1, i, i+1$, when the wind is positive (i.e., when $f'(u) \geq 0$ for the scalar case, or when the corresponding eigenvalue is positive for the system case with a local characteristic decomposition). This numerical flux $\hat{f}_{i+1/2}$ is written as a convex combination of two second order numerical fluxes based on two different substencils of two points each, and the combination coefficients depend on a "smoothness indicator" measuring the smoothness of the solution in each substencil. The detailed formula is

$$\hat{f}_{i+1/2} = w_0 \left[\frac{1}{2} f(u_i) + \frac{1}{2} f(u_{i+1})\right] + w_1 \left[-\frac{1}{2} f(u_{i-1}) + \frac{3}{2} f(u_i)\right], \tag{3}$$

where

$$w_r = \frac{\alpha_r}{\alpha_1 + \alpha_2}, \qquad \alpha_r = \frac{d_r}{(\epsilon + \beta_r)^2}, \qquad r = 0, 1. \tag{4}$$