JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **25**, No. 1, pp. 66-78

doi: 10.4208/jpde.v25.n1.5
February 2012

Boundary Layers Associated with a Coupled Navier-Stokes/Allem-Cahn System: The Non-Characteristic Boundary Case

XIE Xiaoqiang*

School of Science, Shanghai Second Polytechnic University, Shanghai 201209, China.

Received 6 July 2011; Accepted 24 November 2011

Abstract. The goal of this article is to study the boundary layer of Navier-Stokes/Allen-Cahn system in a channel at small viscosity. We prove that there exists a boundary layer at the outlet (down-wind) of thickness ν, where ν is the kinematic viscosity. The convergence in L^2 of the solutions of the Navier-Stokes/Allen-Cahn equations to that of the Euler/Allen-Cahn equations at the vanishing viscosity was established. In two dimensional case we are able to derive the physically relevant uniform in space and time estimates, which is derived by the idea of better control on the tangential derivative and the use of an anisotropic Sobolve imbedding.

AMS Subject Classifications: 35Q35, 35K55, 76D05

Chinese Library Classifications: O175.29

Key Words: Boundary layers; Navier-Stokes; Euler equations; Allen-Cahn; vanishing viscosity limit.

1 Introduction

Consider the incompressible Navier-Stokes/Allen-Cahn system of the form

$$\begin{cases} \frac{\partial}{\partial t}\overrightarrow{u}^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)\overrightarrow{u}^{\nu}+\nabla p^{\nu}=\nu\triangle\overrightarrow{u}^{\nu}-\lambda\nabla\cdot(\nabla v^{\nu}\otimes\nabla v^{\nu}),\\ \nabla\cdot\overrightarrow{u}^{\nu}=0,\\ \frac{\partial}{\partial t}v^{\nu}+(\overrightarrow{u}^{\nu}\cdot\nabla)v^{\nu}=\gamma(\triangle v^{\nu}-f(v^{\nu})), \end{cases} \tag{1.1}$$

where $\overrightarrow{u}^{\nu}$ is the velocity field, v^{ν} and p^{ν} denote the phase function and pressure, ν, λ, γ and ε are positive constants, representing the kinematic viscosity, the surface tension, the

*Corresponding author. *Email address:* xiaoqiangxie021@gmail.com (X. Xie),

mobility and the width of the interface, respectively, and $f(x)=4(x^3-x)/\varepsilon^3$. System (1.1) can be viewed as a phase field model, which describes the motion of a mixture of two incompressible viscous fluids with the same density and viscosity (see [1])

We are interested in the asymptotic behavior of Navier-Stokes/Allen-Cahn equations in the channel $\Omega=(0,L_1)\times(0,L_2)\times(0,h)$ at small viscosity with fluids pumped into the channel at the top ($z=h$) and sucked out the channel at the bottom ($z=0$), the diffused interface has no slip on the boundary, i.e., $\overrightarrow{u}^\nu=(0,0,-U)$, $v^\nu=0$ ($U=\text{constant}>0$) at $z=0$ and $z=h$. Hence in this case the boundary which is permeable is non characteristic, i.e., it is not a stream surface. Throughout this article, we assume that all functions are periodic in x (with period L_1) and in y (with period L_2).

With regular initial data, $\overrightarrow{u}^\nu=\overrightarrow{u}_0$, $v^\nu=v_0$ at $t=0$, and compatibility conditions, strong solutions globally exist in 2D, but locally in 3D in short time (see for instance [2,3]).

At the limit case, namely $\nu=0$, Navier-Stokes/Allen-Cahn system formally becomes the following Euler/Allen-Cahn system:

$$\begin{cases}\dfrac{\partial}{\partial t}\overrightarrow{u}^0+(\overrightarrow{u}^0\cdot\nabla)\overrightarrow{u}^0+\nabla p^0=-\lambda\nabla\cdot(\nabla v^0\otimes\nabla v^0),\\ \nabla\cdot\overrightarrow{u}^0=0,\\ \dfrac{\partial}{\partial t}v^0+(\overrightarrow{u}^0\cdot\nabla)v^0=\gamma(\triangle v^0-f(v^0)),\end{cases}\tag{1.2}$$

with initial data

$$\overrightarrow{u}^0=\overrightarrow{u}_0,\quad v^0=v_0\qquad \text{at } t=0,\tag{1.3}$$

and boundary conditions:

$$\overrightarrow{u}^0=(0,0-U)\text{ and }v^0=0\text{ at }z=h,\quad u_3^0=-U\text{ and }v^0=0\text{ at }z=0.\tag{1.4}$$

We can not expect a convergence result of $\overrightarrow{u}^\nu$ to $\overrightarrow{u}^0$ in the uniform space since they do no have the same traces on the boundary.

The purpose of this paper is to present explicit boundary layer analysis of the Navier-Stokes/Allen-Cahn equations in the case when the boundary is non-characteristic. Our boundary layer analysis is performed in both H^1 space and physically more appealing uniform space. As a consequence we proved that the solutions of Navier-Stokes/Allen-Cahn equations can be approximated by the that of the Euler/Allen-Cahn equations uniformly away from the boundary. Our convergence rate is better controlled than the same results of the Navier-Stokes equations obtained in [4].

Our motivation is the study of the boundary layer associated with incompressible flows and the related question of vanishing viscosity (see for instance [4–17] and among many others).

The article is organized as follows. In Section 2, we show how to choose and construct the correctors for the Navier-Stokes/Allen-Cahn system. In Section 3, we present the short time result on the fully nonlinear case. The final section is devoted to L^∞ boundary layer analysis to the nonlinear case in two dimensional space.