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# Gaussian Curvature Estimates for the Convex Level Sets of Solutions for Some Nonlinear Elliptic Partial Differential Equations

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**Abstract.** We give lower bound estimates for the Gaussian curvature of convex level sets of minimal surfaces and the solutions to semilinear elliptic equations in terms of the norm of boundary gradient and the Gaussian curvature of the boundary.

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**Key Words**: Gaussian curvature estimates; convex level sets; minimal surface; semilinear elliptic equation; maximum principle.

## 1 Introduction

This paper is the continuation of Ma-Ou-Zhang [1]. In [1], they studied the Gaussian curvature estimates of the convex level sets of $p$-harmonic functions defined on convex ring in $\mathbb{R}^n$ with homogeneous Dirichlet boundary conditions. Utilizing the similar technique as in [1], in this paper we study the minimal surface equation and some semilinear elliptic equations.

For minimal surface equation we find a sharp auxiliary function involving the Gaussian curvature of the convex level sets. It is a harmonic function in 2-dimensional case. In higher dimensions, it is a superharmonic function modulo the gradient terms with locally bounded coefficients. Here we use the Laplace-Beltrami operator on the minimal surface (see Gilbarg-Trudinger [2]).

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For semilinear elliptic equation with suitable structure conditions, we find a similar auxiliary function such that it is a superharmonic function in domain modulo the gradient terms with locally bounded coefficients. From these results, we can get the Gaussian curvature estimates of the convex level sets with the norm of boundary gradient and the Gaussian curvature of the boundary. For example we obtain the lower bound estimates for the Gaussian curvature of the level sets of the solutions for a class of semilinear elliptic equations, the strict convexity of its level sets was obtained by Caffarelli-Spruck [3].

The geometry of the level sets of the solutions of elliptic partial differential equations has been extensively studied for a long time. For instance, Ahlfors [4] contains the well-known result that level curves of Green function on simply connected convex domain in the plane are the convex Jordan curves. In 1931, Gergen [5] proved the star-shapeness of the level sets of Green function on 3-dimensional star-shaped domain. In 1956, Shiffman [6] studied the level sets of minimal surface in $\mathbb{R}^3$. In 1957, Gabriel [7] proved that the level sets of the Green function on a 3-dimensional bounded convex domain are strictly convex. Lewis [8] extended Gabriel's result to $p$-harmonic functions in high dimensions. Caffarelli-Spruck [3] generalized Lewis' results to a class of semilinear elliptic partial differential equations. Motivated by the result of Caffarelli-Friedman [9], Korevaar [10] gave a new proof on the results of Gabriel [7] and Lewis [8] using the following observation: if the level sets of the $p$-harmonic function is convex with respect to the gradient direction $\nabla u$, then the rank of the second fundamental form of the level sets is constant. A survey of this subject is given by Kawohl [11]. For more recent related extensions, please see the papers [12–14].

Now we turn to the results of quantitative feature, that is, curvature estimates of the level sets of the solutions to such elliptic problems. For 2-dimensional harmonic function with convex level curves, Longinetti [15], Ortel-Schneider [16], Talenti [17] proved that the curvature of the level curves attains its minimum on the boundary. Longinetti [18] also studied the precisely relation between the curvature of the convex level curves and the height of 2-dimensional minimal surface. Recently, Ma-Ou-Zhang [1] and Chang-Ma-Yang [19] got the Gaussian curvature and principal curvature estimates of the convex level sets for high dimensional harmonic functions, and the estimates give a new approach to get the convexity of the level sets of harmonic functions. In the paper Guan-Xu [20], they obtained a lower bound estimates of principal curvature for level sets of solutions to a class of fully nonlinear elliptic equations under the general structure condition (introduced by Bianchini-Longinetti-Salani [13]) via the approach of constant rank theorem. For more related results, we refer to [21–26] and the references therein.

Now we state our main theorems.

**Theorem 1.1.** *Let $\Omega$ be a smooth bounded domain in $\mathbb{R}^n (n\geq 2)$. Let $u\in C^4(\Omega)\cap C^2(\bar{\Omega})$ be the solution of the following minimal surface equation*

$$\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)=0, \qquad \text{in } \Omega\subset\mathbb{R}^n. \tag{1.1}$$