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Existence of Solutions for Schrödinger-Poisson Systems with Sign-Changing Weight

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Abstract. We study the existence of solutions for the Schrödinger-Poisson system

$$\begin{cases} -\Delta u+u+k(x)\phi u=a(x)|u|^{p-1}u, & \text{in } \mathbb{R}^3, \\ -\Delta\phi=k(x)u^2, & \text{in } \mathbb{R}^3, \end{cases}$$

where $3\le p<5$, $a(x)$ is a sign-changing function such that both the supports of a^+ and a^- may have infinite measure. We show that the problem has at least one nontrivial solution under some assumptions.

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Key Words: Schrödinger-Poisson system; existence result; sign-changing weight.

1 Introduction

In this paper, we are concerned with the existence of solutions for the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u+u+k(x)\phi u=a(x)|u|^{p-1}u & \text{in } \mathbb{R}^3, \\ -\Delta\phi=k(x)u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $3\le p<5$, $a(x)\in C(\mathbb{R}^3,\mathbb{R})$ is a bounded sign-changing continuous function, $k(x)\in L^2(\mathbb{R}^3)$ is a nonnegative function. Set $a(x)=a^+(x)-a^-(x)$ and denote $\Omega^+=\{x\in\mathbb{R}^3: a(x)>0\}$ and $\Omega^-=\{x\in\mathbb{R}^3: a(x)<0\}$. We assume that both Ω^+ and Ω^- have infinite measure.

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Schrödinger-Poisson systems have been widely investigated and it is well known that they have a strongly physical meaning in quantum mechanics and semiconductor theory. During the past few years, many researches have been devoted to studying the existence results of problem (1.1) or its variant, see [1–6]. As we will see in Section 2, problem (1.1) can be easily transformed into a nonlinear Schrödinger equation with a non-local term. Briefly, the Poisson equation can be solved by using Riesz Representation Theorem. That is, for every fixed $u\in H^1(\mathbb{R}^3)$, there is a unique $\phi_u\in\mathcal{D}^{1,2}(\mathbb{R}^3)$ such that $-\Delta\phi_u=k(x)u^2$. Inserting ϕ_u into the first equation gives

$$-\Delta u+u+k(x)\phi_u u=a(x)|u|^{p-1}u,\qquad \text{in } \mathbb{R}^3. \tag{1.2}$$

It is easy to see that problem (1.2) is variational and its solutions correspond to the critical points of the functional defined in $H^1(\mathbb{R}^3)$ by

$$I(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}^3}k(x)\phi_u u^2\mathrm{d}x-\frac{1}{p+1}\int_{\mathbb{R}^3}a(x)|u|^{p+1}\mathrm{d}x. \tag{1.3}$$

The main difficulty in finding critical points of I lies in two aspects. First, the term $k(x)\phi_u u$ in (1.2) makes it difficult to prove the boundedness of (PS) sequences. Second, the imbedding of $H^1(\Omega^+)\hookrightarrow L^{p+1}(\Omega^+)$ is not compact, so the functional I does not satisfy the (PS) condition. These difficulties can be avoided in some particular case, for example, if $a(x)$ and $k(x)$ are constants, then we can study problem (1.1) in $H^1_r(\mathbb{R}^3)$, which embeds into $L^{p+1}(\mathbb{R}^3)(1<p<5)$ compactly, so the (PS) condition retains again, this was done in [7] and [8].

In order to prove the boundedness of the (PS) sequences, we should mention that the authors in [7] and [8] also assumed that $3\le p<5$. As for $1<p<3$, it is difficult to prove the boundedness of the (PS) sequences. A remarkable result for $1<p<3$ was obtained by Ruiz in [6]. In this paper, the author studied problem (1.1) in a special manifold of $H^1_r(\mathbb{R}^3)$ and proved that the (PS) sequences are bounded on this manifold, therefore, the infimum of I on this manifold is attained. This manifold is a combination of the Nehari manifold and the Pohozaev identity. We note the method in [6] can only be used to the autonomous case, i.e., $a(x)$ and $k(x)$ are constants. As the potential and the coefficient of the nonlinearity are not radial, there are few existence results on Schröginger-Poisson system except a recent paper [3] by Cerami and Vaira. In this paper, the authors assume that $3<p<5$, $\lim_{|x|\to\infty}k(x)=0$ and $\lim_{|x|\to\infty}a(x)=a_\infty$, they proved that the lack of compactness is caused by the solutions of the "limiting equation"

$$-\Delta u+u=a_\infty|u|^{p-1}u,\qquad \text{in } \mathbb{R}^3. \tag{1.4}$$

More precisely, let

$$I_\infty(u)=\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla u|^2+u^2)\mathrm{d}x-\frac{1}{p+1}\int_{\mathbb{R}^3}a_\infty|u|^{p+1}\mathrm{d}x,\qquad u\in H^1(\mathbb{R}^3), \tag{1.5}$$

$$N_\infty=\{u\in H^1(\mathbb{R}^3)\setminus\{0\}:\langle I'_\infty(u),u\rangle=0\}, \tag{1.6}$$