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Existence of Solutions to a Semilinear Elliptic System Through Generalized Orlicz-Sobolev Spaces

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Abstract. This paper is concerned with the existence theory of a semilinear elliptic system. In particular, we will prove that the system has a nontrivial positive solution in some appropriate solution spaces.

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1 Introduction

In this paper we study nonlinear elliptic systems of the type

$$\begin{cases} -\Delta u = f(x,v) & \text{in } \Omega, \\ -\Delta v = g(x,u) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, and Δ is the Laplace operator. $f,g:\Omega\times\mathbb{R}\longrightarrow\mathbb{R}$ are suitable functions satisfying $f(x,0)=g(x,0)=0$. An example included in this study is obtained when $f(x,v)=f(v)$ and $g(x,u)=g(u)$, In this case the system (1.1) is reduced to

$$\begin{cases} -\Delta u = f(v) & \text{in } \Omega, \\ -\Delta v = g(u) & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem of the type (1.2) was studied by Figueiredo et al. in [1] and Clément, et al. in [2]; the authors also proved the existence of positive solution. Moreover, the special

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case for which f and g are pure powers has been studied by many authors of which we cite [3–7]. Indeed, if $f(v)=|v|^{\alpha-1}v$ and $g(u)=|u|^{\beta-1}u$ with $\alpha,\beta>0$ satisfying

$$1>\frac{1}{\alpha+1}+\frac{1}{\beta+1}>1-\frac{2}{n}, \tag{1.3}$$

then (1.2) possesses at least one positive solution for dimensions $n\geq 3$.

The first inequality corresponds to super-linearity, which leads to existence of solutions via a mini-max argument. While the second inequality corresponds to sub-criticality which guarantees the required compactness in the application of a Mountain Pass Theorem as well as regularity of solutions through a bootstrap argument. In this note a similar condition **H** (which will be defined later on) will be used but the real numbers α and β that appears on the right hand side do not need to be the same as the ones in the left hand side.

In this paper, we obtain a positive solution of the system (1.1) by inverting the first equation in (1.1) and using a Mountain Pass Theorem given by Ambrozetti-Rabinowitz (see [3]). The right setting for this approach is the use of Sobolev-generalized Orlicz spaces.

Before we state our main result we have to fix the conditions needed on the functions f and g. For this purpose, we introduce a class of functions namely N-functions and Orlicz indices.

Definition 1.1. *Let (Ω,Σ,μ) be a measure space. A real valued function φ defined on $\Omega\times\mathbb{R}$ will be said N-function if it satisfies the following conditions:*

(i) *$\varphi(x,u)$ is nondecreasing, continuous function of u.*

(ii) *$\varphi(x,-u)=-\varphi(x,u)$, for all $u\in\mathbb{R}$ and $x\in\Omega$.*

(iii) *$\lim_{u\to+\infty}\varphi(x,u)=+\infty$, for all $x\in\Omega$.*

(iv) *$\varphi(x,u)$ is Σ-measurable function of x, for all $u\geq 0$.*

Remark 1.1. Let φ be an N-function. For each $x\in\Omega$, we set

$$\varphi^*(x,v)=\sup\Big\{\tau;\varphi(x,\tau)\leq v\Big\}.$$

It is well known that the function $\varphi^*(x,v)$ is also an N-function.

Notation 1.1. We will associate to an N-function φ, the functions Φ and Φ^* defined as follows:

$$\Phi(x,u)=\int_0^u\varphi(x,s)\mathrm{d}s,\quad \Phi^*(x,u)=\int_0^u\varphi^*(x,s)\mathrm{d}s, \tag{1.4}$$

where Φ^* is termed as the complementary function to Φ in the sense of Young.