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PURE LAGRANGIAN AND SEMI-LAGRANGIAN FINITE ELEMENT METHODS FOR THE NUMERICAL SOLUTION OF CONVECTION-DIFFUSION PROBLEMS

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This paper is dedicated to Professor Francisco J. Lisbona in his 65th birthday

Abstract. In this paper we propose a unified formulation to introduce and analyze (pure) Lagrangian and semi-Lagrangian methods for solving convection-diffusion partial differential equations. This formulation allows us to state classical and new numerical methods. Several examples are given. We combine them with finite element methods for spatial discretization. One of the pure Lagrangian methods we introduce has been analyzed in [4] and [5] where stability and error estimates for time semi-discretized and fully-discretized schemes have been proved. In this paper, we prove new stability estimates. More precisely, we obtain an $l^\infty(H^1)$ stability estimate independent of the diffusion coefficient and, if the underlying flow is incompressible, we get a stability inequality independent of the final time. Finally, the numerical solution of a test problem is presented that confirms the new stability results.

Key words. convection-diffusion equation, pure Lagrangian method, semi-Lagrangian method, Lagrange-Galergin method, characteristics method, second order schemes, finite element method.

1. Introduction

Convection-diffusion equations model a variety of important problems from different fields of engineering and applied sciences. In many cases the diffusive term is much smaller than the convective one and then upwinding has to be introduced in the numerical scheme. This can be done by using characteristics method which are based on time discretization of the material time derivative. These methods were introduced in the beginning of the 1980s combined with finite differences or finite elements for space discretization (see [11], [18]). In this context they are also called Lagrange-Galerkin methods. The classical methods of characteristics are written in Eulerian coordinates and they are related to semi-Lagrangian schemes (see [13]). Lagrange-Galerkin methods have been mathematically analyzed and applied to different problems by several authors. For example, in [23], [18] the classical first order characteristic method combined with finite elements applied to convection-diffusion equations is studied, and in [22], [6] and [7] second order Lagrange-Galerkin methods are analyzed. More precisely, if $\triangle t$ denotes the time step, h the mesh-size and k the degree of the finite elements space, estimates of the form $O(h^k) + O(\triangle t)$ in the $l^\infty(L^2(\mathbb{R}^d))$-norm are shown in [23] (d denotes the dimension of the spatial domain). In [18] error estimates of the form $O(h^k) + O(\triangle t) + O(h^{k+1}/\triangle t)$ in the $l^\infty(L^2(\Omega))$-norm are obtained under the assumption that the normal velocity vanishes on the boundary of Ω. In [22] a second order characteristics method for solving constant

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coefficient convection-diffusion equations with Dirichlet boundary conditions is studied. Stability and $O(\triangle t^2) + O(h^k)$ error estimates in the $l^\infty(L^2(\Omega))$-norm are stated (see also [6] and [7] for further analysis).

Recently, for linear convection diffusion problems, we have introduced the so-called *pure Lagrangian methods* combined with finite elements. They are obtained by discretizing the problem which has been first written in Lagrangian or material coordinates. In particular, in [4] and [5] $l^\infty(H^1(\Omega))$ stability and $l^\infty(H^1(\Omega))$ error estimates of order $O(\Delta t^2) + O(h^k)$ were proved for a second order pure Lagrange-Galerkin method. In [10], semi-Lagrangian and pure Lagrangian methods are proposed and analyzed for convection-diffusion equations. Error estimates for a Galerkin discretization of a pure Lagrangian formulation and for a discontinuous Galerkin discretization of a semi-Lagrangian formulation are obtained. The estimates are written in terms of the projections constructed in [8] and [9]. In [4] and [5] a pure Lagrangian formulation has been used for a more general problem. Specifically, we have considered a (possibly degenerate) variable coefficient diffusive term instead of the simpler Laplacian, general mixed Dirichlet-Robin boundary conditions, and a time dependent domain. Moreover, we have analyzed a scheme with approximate characteristic curves and presented numerical results for pure Lagrangian and semi-Lagrangian methods.

In the present paper, we introduce a unified formulation to state pure Lagrangian and semi-Lagrangian methods for solving linear convection-diffusion equations.

Our approach uses the formalism of continuum mechanics in which classical and new methods can be introduced in a natural way (see for instance [15]).

The paper is organized as follows. In Section 2 the linear convection-diffusion Cauchy problem is posed in a time dependent bounded domain and some hypotheses and notations concerning motions are stated. In Section 3, we introduce a quite general change of variable obtaining a new strong formulation of the linear convection-diffusion Cauchy problem. Moreover, the standard associated weak problem is obtained. In Section 4, pure Lagrangian and semi-Lagrangian schemes are proposed. All these methods arise from the formulation obtained in the previous section. In Section 5, a second order pure Lagrange-Galerkin scheme is proposed for second order approximate characteristics. We recall some properties verified by this method. Moreover, under suitable hypotheses on the data, two new stability results are proved for small enough time step. One of them is independent of the diffusion coefficient and the other one is independent of the final time. In Section 6, in order to check experimentally these stability results, we solve a linear convection-diffusion problem.

2. Statement of the linear convection diffusion problem. General assumptions and notations.

Let Ω be a bounded domain in $\mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary Γ divided into two parts: $\Gamma = \Gamma^D \cup \Gamma^R$, with $\Gamma^D \cap \Gamma^R = \emptyset$. Let T be a positive constant and $X_e : \overline{\Omega} \times [0, T] \longrightarrow \mathbb{R}^d$ be a motion in the sense of Gurtin [15]. In particular, $X_e \in \mathbf{C}^3(\overline{\Omega} \times [0, T])$ and for each fixed $t \in [0, T]$, $X_e(\cdot, t)$ is a one-to-one function satisfying

$$\det F(p, t) > 0 \quad \forall p \in \overline{\Omega}, \tag{1}$$

being $F(\cdot, t)$ the Jacobian tensor of $X_e(\cdot, t)$. We call $\Omega_t = X_e(\Omega, t)$, $\Gamma_t = X_e(\Gamma, t)$, $\Gamma_t^D = X_e(\Gamma^D, t)$ y $\Gamma_t^R = X_e(\Gamma^R, t)$, for $t \in [0, T]$. We assume that $\Omega_0 = \Omega$. Let us