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A PRIORI BOUNDS FOR GLOBAL SOLUTIONS OF HIGHER-ORDER SEMILINEAR PARABOLIC PROBLEMS

Xing Ruixiang and Pan Hongjing
(School of mathematical sciences, Peking University, Beijing 100871, China)
(E-mail: xingruixiang@math.pku.edu.cn & panhj@math.pku.edu.cn)
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Abstract In this paper, we derive a priori bounds for global solutions of 2m-th order semilinear parabolic equations with superlinear and subcritical growth conditions. The proof is obtained by a bootstrap argument and maximal regularity estimates. If $n \geqslant \frac{10}{3}m$, we also give another proof which does not use maximal regularity estimates.
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1. Introduction

In this paper, we consider the higher-order semilinear parabolic problem:

$$u_t + (-\triangle)^m u = |u|^{p-1}u, \qquad x \in \Omega,\ 0 < t < \infty, \tag{1.1}$$

$$u = \frac{\partial}{\partial \nu}u = \cdots = (\frac{\partial}{\partial \nu})^{m-1}u = 0, \qquad x \in \partial\Omega,\ 0 < t < \infty, \tag{1.2}$$

$$u(x,0) = u_0(x) \in H_0^m(\Omega), \tag{1.3}$$

with $1 < p < \frac{n+2m}{n-2m}$ $(n > 2m)$ or $1 < p < \infty$ $(n \leqslant 2m)$, where $\Omega \subset \mathbb{R}^n$ is a smoothly bounded domain and ν is the exterior unit normal vector to $\partial\Omega$. Higher-order semilinear and quasilinear parabolic equations occur in a number of models from physics, mechanics, and biological system. Some important examples of fourth-order equation are the extended Fisher-Kolmogrov equation, the Swift-Hohenberg equation, the Kuramoto-Sivashinsky equation and the equations in thin film theory. We refer to [1] and the references therein for detailed discussions, motivations and various study methods.

The main purpose of this paper is to derive a priori bound for the global solution of the problem (1.1)-(1.3). It is well known that the global solution of the problem is bounded when $m = 1$ (see [2–6]). P. Quittner [6] obtained a uniform bound for the global solution with all superlinear and subcritical p when $m = 1$. But no estimates seem to be available for the higher-order case. We will extend P. Quittner's result for $m = 1$ to arbitrary positive integer m.

The main result of this paper is the following:

Theorem 1.1 *Let u be a solution of* $(1.1)-(1.3)$. *If* $u_0 \in H_0^m(\Omega)$, *then*

$$\sup_{t \geqslant \delta_0} \|u(t)\|_{C^{2m-1}(\overline{\Omega})} \leqslant C,$$

where C depends only on $\delta_0, \|u_0\|_{H^m(\Omega)}, n, p$ *and* Ω.

In our proof, we adapt the approaches proposed in [6]. The key ingredients are a bootstrap argument and maximal regularity estimates. If $n \geqslant \frac{10}{3}m$, we also give another proof without using maximal regularity estimates. Therefore, the second method can be applied to the problems where the maximal regularity is not known or does not hold.

Such a type of a priori bounds plays a crucial role in many applications. Let us mention several examples. The first one is concerned with deriving blow-up rates for some parabolic problems via a priori bounds for suitable equations (see [7, 8]). The second application is to show the existence of sign changing stationary solutions by the dynamical method (see [9]). The third one is concerned with establishing Deformation Lemmas in variational problems instead the gradient by the heat flow (see [10]).

This paper is organized as follows. In Section 2, we describe the corresponding analytic semigroup theory and a variation-of-constants formula. The local existence and regularity and some auxiliary lemmas are also shown. Section 3 is about some integral estimates. In Section 4, the proof of the main theorem is completed by a bootstrap argument combining with the maximal regularity estimates. Lastly in Section 5, we present another proof without using the maximal regularity, but we need the stronger condition on n. At the end of the paper, we extend the main result to the more general problem where $|u|^{p-1}u$ in (1.1) is replaced by $f(x, u)$.

2. Local Existence, Regularity and Some Lemmas

In this section, we describe the corresponding analytic semigroup theory and present a variation-of-constants formula. The formula is a basis to get suitable existence and regularity properties of solutions. In addition, some useful lemmas will be given.

Let $\mathcal{A}_q$ denote $(-\triangle)^m$ and $\mathcal{D}(\mathcal{A}_q) = \{u \in W^{2m,q}(\Omega) : u = \frac{\partial}{\partial\nu}u = \cdots = (\frac{\partial}{\partial\nu})^{m-1}u = 0 \text{ on } \partial\Omega\}$.

It is well known that for $q \in (1, +\infty)$, there exist $\omega_q > 0$ and $M_q > 0$ such that

$$\sum_{k=0}^{2m} |\lambda|^{1-\frac{k}{2m}} \Big(\sum_{|\alpha|=k} \|D^\alpha u\|_{L^q(\Omega)} \Big) \leqslant M_q \|\lambda - \mathcal{A}_q\|_{L^q(\Omega)},$$