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GLOBAL L^p ESTIMATES FOR THE PARABOLIC EQUATION OF THE BI-HARMONIC TYPE*

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Abstract In this paper similarly to the second-order case, we give an elementary and straightforward proof of global L^p estimates for the initial-value parabolic problem of the bi-harmonic type. Moreover, we obtain the existence and uniqueness of the solutions in the suitable space using the potential theory, Marcinkiewicz interpolation theorem and approximation argument. Meanwhile, by the same approach we can deal with the general polyharmonic cases.

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1. Introduction

Schauder estimates and L^p estimates are two fundamental estimates for elliptic and parabolic equations. They play important roles in the theory of partial differential equations, and are the basis for the existence, uniqueness and regularity of solutions. Schauder estimates and L^p estimates for the second order parabolic problems have been obtained by different techniques (see [1–6]). Many authors [7–9] have obtained Schauder estimates for the general linear high-order parabolic boundary value problem with the help of fundamental solutions and Green functions. Moreover, in [7] Solonnikov also obtained L^p estimates for the general linear high-order parabolic problem using the same approach as Schauder estimates. In this paper similarly to the second-order case, we get global L^p estimates for the fourth-order parabolic Cauchy problem of the bi-harmonic type. We first obtain a formal expression of solutions by the Fourier transform and then get the regularity, uniqueness, existence of solutions using the potential theory,

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Marcinkiewicz interpolation theorem and approximation argument. Our approach is much simpler and more straightforward. Meanwhile, by the same approach one may deal with the general polyharmonic cases (see Remark 1.3).

In this paper we mainly consider global L^p estimates of the following fourth-order parabolic problem

$$u_t + \triangle^2 u = f \qquad \text{in } \mathbb{R}^n \times (0,T], \tag{1.1}$$

$$u|_{t=0} = 0 \qquad \text{in } \mathbb{R}^n. \tag{1.2}$$

The equation (1.1) is the parabolic equation of the bi-harmonic type in the sense that the elliptic term is a double Laplacian Δ^2. It is a simple form of the Cahn-Hilliard equation

$$u_t + \kappa\Delta(\Delta u - W'(u)) = f,$$

which was originally proposed for modelling phase separation phenomena in a binary mixture (see [10]). The generalized Cahn-Hilliard equation

$$u_t + \mathrm{div}[M(u)\nabla(K\Delta u - W'(u)] = 0,$$

has become a pillar of materials science and engineering (see [11]). Many people have studied this equation for different aspects (see [12,13]). In [14,15] the following equation

$$u_t + \lambda\Delta^2 u = \gamma \mathrm{div}(u^m \nabla u) \ \text{ in } \mathbb{R}^n \times (0,\infty) \tag{1.3}$$

is used to model the long range effect of insects dispersal. When $\lambda = 1$ and $\gamma = 0$, the equation (1.3) is simplified to be the equation (1.1). The following generalized thin film equation

$$u_t + \mathrm{div}(|\nabla\triangle u|^{p-2}\nabla\triangle u) = f \qquad \text{for } 1 < p < \infty, \tag{1.4}$$

which originated from the modelling of the thin film, has been studied extensively (see [16, 17]). When $p = 2$, the equation (1.4) reads as the equation (1.1).

Let $u(x,t) : \mathbb{R}^n \times (0,T] \to \mathbb{R}$ be a function. We denote its weak partial derivative

$$D_x^\nu u(x,t) = \frac{\partial^{|\nu|} u(x,t)}{\partial x_1^{\nu_1} \cdots \partial x_n^{\nu_n}},$$

where $\nu = (\nu_1, \nu_2, ..., \nu_n)$ is a multiple index, $\nu_i \geq 0$ $(i = 1, 2, ..., n)$ and $|\nu| = \sum_{i=1}^n \nu_i$. For simplicity, we often omit the subscript x in $D_x^\nu u$ and denote $D^k u = \{D^\nu u : |\nu| = k\}$. Let $\mathbb{R}_+ = [0,\infty)$, and $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times \mathbb{R}_+$. Moreover, we give the definition of an anisotropic Sobolev space.

Definition 1.1 *Let $p > 1$. We say that $u \in W_p^{4,1}(\mathbb{R}^n \times (0,T])$ if*

$$\|u\|_{W_p^{4,1}(\mathbb{R}^n\times(0,T])} =: \left\{ \|u_t\|^p_{L^p(\mathbb{R}^n\times(0,T])} + \sum_{0\leq|\nu|\leq 4} \|D^\nu u\|^p_{L^p(\mathbb{R}^n\times(0,T])} \right\}^{1/p} < \infty.$$