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BESOV SPACES AND SELF-SIMILAR SOLUTIONS FOR NONLINEAR EVOLUTION EQUATIONS*

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Dedicated to Professor Jiang Lishang on the occasion of his 70th birthday

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Abstract In this paper, we establish the existence of global self-similar solutions for the heat and convection-diffusion equations. This we do in some homogeneous Besov spaces using the theory of Besov spaces and the Strichartz estimates. Further, the structure of the self-similar solutions has also been established by using an equivalent norm for Besov spaces.

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1. Introduction

In this paper we study the existence and regularity of global self-similar solutions of the Cauchy problem for the semi-linear heat equation

$$u_t - \triangle u = \mu u^{\alpha+1}, \quad u(0,x) = f(x) \tag{1.1}$$

and the Cauchy problem for the convection-diffusion equation

$$\partial_t u - \triangle u = \vec{a} \cdot \nabla(|u|^\alpha u), \qquad u(0,x) = f(x), \tag{1.2}$$

where $\mu \in \mathbb{R}$, $\vec{a} \in \mathbb{R}^n \setminus \{0\}$, $\alpha > 0$, $u = u(t,x)$ is a real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^n$ and the initial data f is a real-valued function.

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Self-similar solutions have been studied for other semilinear evolution equations such as the semilinear wave equation [1-4], the Navier-Stokes equations [5, 6] and the Schroedinger equations [7-10]. They often describe the large time behavior of general global solutions to the evolution equations under certain conditions. For example, it was shown in [6] that self-similar solutions for the Navier-Stokes equations constructed by Cannone [5] provide the large time asymptotic behavior of the global solutions.

A solution $u(t,x)$ of (1.1) or (1.2) is called a self-similar solution if for $\lambda > 0$,

$$u(t,x) = \lambda^{\frac{2}{\alpha}} u(\lambda^2 t, \lambda x).$$

It is easy to verify that u is a self-similar solution if and only if

$$u(t,x) = t^{-\frac{1}{\alpha}} u\left(1, \frac{x}{\sqrt{t}}\right) = t^{-\frac{1}{\alpha}} V\left(\frac{x}{\sqrt{t}}\right)$$

for some function $V(x)$ called the profile of the self-similar solution u. Thus the Self-similar solution to nonlinear evolution equations can be studied through the study of the associated semi-linear elliptic equations for $V(x)$. However, it is usually very difficult to solve such nonlinear elliptic equations. On the other hand, the initial data for self-similar solutions must satisfy, for $\lambda > 0$,

$$f(x) = \lambda^{\frac{2}{\alpha}} f(\lambda x). \tag{1.3}$$

This leads to another way of looking for self-similar solutions of (1.1) or (1.2) by the study of small global well-posedness in some suitable function spaces of the Cauchy problem (1.1) or (1.2) with initial data f satisfying (1.3). These new global solutions admit a class of self-similar solutions. However, the condition (1.3) means that f is homogeneous degree $-2/\alpha$. Such homogeneous functions, in general, do not belong to the usual spaces such as the usual Sobolev space $H^{s,p}$, where the global well-posedness of the Cauchy problem holds. Thus, in order to construct self-similar solutions for evolution equations such as (1.1) or (1.2) we have to choose a suitable homogeneous Banach space X of degree $-2/\alpha$ and to show that the problem generates a global flow in X.

The well-posedness of the Cauchy problem for the heat equation (1.1) has been studied by many authors. For example, the existence and uniqueness of solutions have been studied in [7, 11-16] for the case when the initial data is in Sobolev spaces and in [17] for the case when the initial data is in Besov spaces. Self-similar solutions have also been dealt with for the heat equation (1.1) in [18, 14] by the study of the associated elliptic problem and in, e.g. [7] by studying the Cauchy problem. In [19, 20], the global solutions of the nonlinear heat equation have been shown to be asymptotically close to its self-similar solution. On the other hand, the global well-posedness including the large time behavior of the solution has been proved for the convection-diffusion (1.2) in [21], whilst the existence of positive self-similar solutions for (1.2) has been established in [22] in the case when $\alpha = 1/n$ through the study of the associated elliptic problem.