J. Partial Diff. Eqs. **19**(2006), 208–217
©International Academic Publishers Vol.19 No.3

SOME NONLINEAR ELLIPTIC EQUATIONS HAVE ONLY CONSTANT SOLUTIONS*

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Dedicated to K. C. Chang with high esteem and warm friendship

(Received Apr. 11, 2006)

Abstract We study some nonlinear elliptic equations on compact Riemannian manifolds. Our main concern is to find conditions which imply that such equations admit only constant solutions.

Key Words Nonlinear elliptic equations; constant solutions.
2000 MR Subject Classification 35J60.
Chinese Library Classification O175.29, O175.25.

1. Introduction

Motivated by some recent results and questions raised in [1], we study some nonlinear elliptic equations of the form

$$\begin{cases} -\Delta_g u = f(u) & \text{on } M, \\ u > 0 & \text{on } M, \end{cases} \tag{1.1}$$

where (M, g) is a compact Riemannian manifold of dimension $n \geq 2$, without boundary, and $f : (0, +\infty) \to \mathbb{R}$ is a smooth function. Our main concern is to find conditions on M and f which imply that (1.1) admits only constant solutions.

We will present results in two directions:

1) The case where $M = S^n, n \geq 3$, equipped with its standard metric g_0

*The first author (H.B) is also a member of the Institut Universitaire de France and his work is partially supported by an EC Grant through the RTN Program "Front-Singularities" HPRN-CT-2002-00274. The second author (Y. L.) is partially supported by NSF Grant DMS-0401118.

In this case our first result is

Theorem 1 *Assume that* $(M,g) = (S^n, g_0), n \geq 3$, *and*

$$h(t) := t^{-\frac{n+2}{n-2}} \left(f(t) + \frac{n(n-2)}{4} t \right) \text{ is decreasing on } (0,\infty). \tag{1.2}$$

Then any solution of (1.1) *is constant.*

A typical example is the case

$$f(t) = t^p - \lambda t,\ p > 1, \lambda > 0, \tag{1.3}$$

so that (1.1) becomes

$$\begin{cases} -\Delta_g u = u^p - \lambda u & \text{on } S^n, \\ \quad u > 0 & \text{on } S^n. \end{cases} \tag{1.4}$$

Corollary 1 *Assume that* $p \leq (n+2)/(n-2)$ *and* $\lambda \leq n(n-2)/4$, *and at least one of these inequalities is strict. Then the only solution of* (1.4) *is the constant* $u = \lambda^{1/(p-1)}$.

In fact, Corollary 1 is originally due to Gidas-Spruck [2]. But our argument is quite different from theirs; they rely on some remarkable identities while our method uses moving planes.

When $p = (n+2)/(n-2)$ the conclusion of Corollary 1 is sharp. Indeed if $\lambda = n(n-2)/4$ there is a well-known family of nonconstant solutions; moreover all solutions of (1.4) belong to this family. However when $p < (n+2)/(n-2)$, B. Gidas and J. Spruck established a better result which was later sharpened by M.F. Bidaut-Veron and L. Veron. Namely they proved

Theorem 2([2],[3]) *Assume that* $p < (n+2)/(n-2)$ *and* $\lambda \leq n/(p-1)$. *Then the only solution of* (1.4) *is the constant* $u = \lambda^{1/(p-1)}$.

Remark 1 The proof of Theorem 2 in [2] and [3] is based on some remarkable identities. Our proof of Theorem 1 uses the method of moving planes. It would be very interesting to find a proof of Theorem 2 based on moving planes.

On the other hand, bifurcation analysis (see [3] and Section 4 below) yields

Theorem 3 *Assume* $p < (n+2)/(n-2)$ *and* $\lambda > n/(p-1)$ *with* $|\lambda - n/(p-1)|$ *small. Then there exist nonconstant solutions of* (1.4).

Remark 2 When $p > \frac{n+2}{n-2}$, there exist nonconstant solutions of (1.4) for some values of $\lambda < \frac{n(n-2)}{4}$. Indeed bifurcation theory (see Section 4 and Remark 7 there) implies the existence of a branch of nonconstant solutions emanating from the constant solutions at the value $\lambda = \frac{\nu}{p-1}$ where $\nu = n$ is the second eigenvalue of $-\Delta_{g_0}$ on S^n; note that $\frac{\nu}{p-1} < \frac{n(n-2)}{4}$ since $p > \frac{n+2}{n-2}$. These solutions exist for $\lambda < \frac{\nu}{p-1}$ and $|\lambda - \frac{\nu}{p-1}|$ sufficiently small.

Open Problem 1 When $p > \frac{n+2}{n-2}$, we do not know any result asserting that for some value of $\lambda > 0, \lambda$ small, equation (1.4) admits only the constant solution