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PARAMETER-UNIFORM CONVERGENCE FOR A FINITE DIFFERENCE METHOD FOR A SINGULARLY PERTURBED LINEAR REACTION-DIFFUSION SYSTEM WITH DISCONTINUOUS SOURCE TERMS

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This paper is dedicated to Professor Francisco J. Lisbona on his 65th birthday

Abstract. A singularly perturbed linear system of second order ordinary differential equations of reaction-diffusion type with discontinuous source terms is considered. A small positive parameter multiplies the leading term of each equation. These singular perturbation parameters are assumed to be distinct. The components of the solution exhibit overlapping boundary and interior layers. A numerical method is constructed that uses a classical finite difference scheme on a piecewise uniform Shishkin mesh. It is proved that the numerical approximations obtained by this method are essentially first order convergent uniformly with respect to all of the perturbation parameters. Numerical illustrations are presented in support of the theory.

Key words. Singular perturbation problems, system of differential equations, reaction - diffusion equations, discontinuous source terms, overlapping boundary and interior layers, classical finite difference scheme, Shishkin mesh, parameter - uniform convergence.

1. Introduction

A singularly perturbed linear system of second order ordinary differential equations of reaction - diffusion type with discontinuous source terms is considered in the interval $\Omega = \{x : 0 < x < 1\}$. A single discontinuity in the source terms is assumed to occur at a point $d \in \Omega$. Introduce the notation $\Omega^- = (0, d)$, $\overline{\Omega^-} = [0, d]$, $\Omega^+ = (d, 1)$, $\overline{\Omega^+} = [d, 1]$ and denote the jump at d in any function $\vec{\omega}$ by $[\vec{\omega}](d) = \vec{\omega}(d+) - \vec{\omega}(d-)$. The corresponding self-adjoint two point boundary value problem is

$$(1)\quad -E\vec{u}''(x) + A(x)\vec{u}(x) = \vec{f}(x) \text{ on } \Omega^- \cup \Omega^+,\ \vec{u} \text{ given on } \Gamma \text{ and } \vec{f}(d-) \neq \vec{f}(d+)$$

where $\Gamma = \{0, 1\}$, $\overline{\Omega} = \Omega \cup \Gamma$. The norms $\| \vec{V} \| = \max_{1\leq k\leq n} |V_k|$ for any n-vector $\vec{V}$, $\| y \| = \sup_{0\leq x\leq 1} |y(x)|$ for any scalar-valued function y and $\| \vec{y} \| = \max_{1\leq k\leq n} \| y_k \|$ for any vector-valued function $\vec{y}$ are introduced. Here $\vec{u}$ is a column $n-$vector, E and $A(x)$ are $n\times n$ matrices, $E = \mathrm{diag}(\vec{\varepsilon})$, $\vec{\varepsilon} = (\varepsilon_1, \cdots, \varepsilon_n)$ with $0 < \varepsilon_i \leq 1$ for all $i = 1, \ldots, n$. The ε_i are assumed to be distinct and, for convenience, to have the ordering

$$\varepsilon_1 < \cdots < \varepsilon_n.$$

For simplicity, cases with some of the parameters coincident are not considered here. In these cases the number of layer functions is reduced and, consequently, the number of transition parameters in the Shishkin mesh defined in Section 4 is reduced. The methods of proof are essentially the same.

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The problem can also be written in the operator form

$$\vec{L}\vec{u} \;=\; \vec{f} \text{ on } \Omega^- \cup \Omega^+,\ \vec{u} \text{ given on } \Gamma \text{ and } \ \vec{f}(d-) \neq \vec{f}(d+)$$

where the operator $\vec{L}$ is defined by

$$\vec{L} \;=\; -ED^2 + A, \quad D^2 = \frac{d^2}{dx^2}.$$

For all $x \in \overline{\Omega}$, it is assumed that the components $a_{ij}(x)$ of $A(x)$ satisfy the inequalities

$$a_{ii}(x) > \sum_{\substack{j\neq i\\ j=1}}^{n} |a_{ij}(x)| \ \text{ for } \ 1 \le i \le n \ \text{ and } \ a_{ij}(x) \le 0 \ \text{ for } \ i \neq j \tag{2}$$

and, for some α,

$$0 < \alpha < \min_{\substack{x\in[0,1]\\ 1\le i\le n}} (\sum_{j=1}^{n} a_{ij}(x)). \tag{3}$$

It is assumed that $a_{ij} \in C^{(2)}(\overline{\Omega})$, $f_i \in C^{(2)}(\Omega^- \cup \Omega^+)$ for $i, j = 1, \ldots, n$. Then (1) has a solution $\vec{u} \in C(\overline{\Omega}) \cap C^{(1)}(\Omega) \cap C^{(4)}(\Omega^- \cup \Omega^+)$. Because $\vec{f}$ is discontinuous at d, the solution $\vec{u}(x)$ does not necessarily have a continuous second order derivative at the point d. Thus $\vec{u}(x) \notin C^2(\Omega)$, but the first derivative of the solution exists and is continuous. In Section 2, for the construction of the solution in Theorem 2.1 and in the definition of the singular component, we need the Schur product of two $n-$ vectors, which is defined by

$$\vec{\mu} \cdot \vec{\eta} \;=\; (\mu_1\eta_1,\, \mu_2\eta_2,\, \ldots,\, \mu_n\eta_n) \;\in\; \mathbb{R}^n \tag{4}$$

for $\vec{\mu} = (\mu_1,\, \mu_2,\, \ldots,\, \mu_n)$ and $\vec{\eta} = (\eta_1,\, \eta_2,\, \ldots,\, \eta_n) \in \mathbb{R}^n$. It is also assumed that

$$\sqrt{\varepsilon}_n \le \frac{\sqrt{\alpha}}{6}. \tag{5}$$

Throughout the paper C denotes a generic positive constant, which is independent of x and of all singular perturbation and discretization parameters. Furthermore, inequalities between vectors are understood in the componentwise sense.

For a general introduction to parameter-uniform numerical methods for singular perturbation problems, see for example [1], [2] and [3]. Parameter-uniform numerical methods for scalar problems with discontinuous data are reported in [4], [5], [6] and [7]. The present paper extends the results in [4] for a single equation to a general system of equations.

The plan of the paper is as follows. In the next two sections, the analytical results of the continuous problem are given. In Section 4 piecewise-uniform Shishkin meshes, which are fitted to resolve the interior and boundary layers, are introduced. In Section 5 the discrete problem is defined and the corresponding maximum principle and stability result are established. In Section 6 the statement and proof of the parameter-uniform error estimate are given. Section 7 contains numerical illustrations.