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ON SOME FORWARD-BACKWARD DIFFUSION EQUATIONS

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Dedicated to Professor K. C. Chang on his seventieth birthday

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Abstract We describe an application of the partial differential inclusion method to the existence of Lipschitz weak solutions for certain forward-backward diffusion equations including the one-dimensional version of the Perona-Malik equation in image processing.

Key Words forward-backward diffusion equation, partial differential inclusion, relaxation property, existence of weak solutions, image processing.

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In some earlier works of Professor Chang, elliptic equations with discontinuous nonlinearities were studied and set-valued mappings are used to solve such equations (see, for example [1-7] In this talk we use the partial differential inclusion method for set-valued mappings to study mainly the existence problem of the following one-dimensional forward-backward diffusion equation under the homogeneous natural boundary condition:

$$\begin{cases} u_t = a(u_x)_x, & (t,x) \in (0,T) \times (0,\, 1) := Q_T, \\ u(0,x) = u_0(x), & 0 \le x \le l, \\ a(u_x(t,0)) = a(u_x(t,l)) = 0, & 0 \le t \le T, \end{cases} \tag{1}$$

where the flux function $a(s)$ is not monotone. There are many examples of such non-monotone $a(\cdot)$ arising from applied mathematics such as

Model 1) the general coercive forward-backward diffusion model [8] where $a(s)s > c|s|^2$ while $a(\cdot)$ is not monotone,

Model 2) the well-known one-dimensional Perona-Malik model where $a(s) = s/(1+s^2)$ in image processing for edge detection [9-13],

Model 3) the simplest double well model in the study of microstructure $a(s) = f'(s)$ with $f(s) = \mathrm{dist}^2(s, \{-1, 1\})$ the squared distance function and its generalizations under lower order perturbations [14, 15].

There are some other related models such as

Model 4) the fast diffusion model where $a(s) = \mathrm{sign}(s)\log|s|$ [16]. We may consider the Cauchy problem or the initial boundary value problem under the Dirichlet condition.

$$\begin{cases} u_t = a(u_x)_x, & (t,x) \in (0,T)\times \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R} \end{cases} \tag{2}$$

Model 5) the two-dimensional model based on the Perona-Malik numerical scheme [9, 12]:

$$\begin{cases} u_t = a(u_x)_x + a(u_y)_y, & (t,x,y) \in (0,T)\times D := Q_T, \\ u(0,x,y) = u_0(x,y), & (x,y) \in \bar{D}, \\ \mathbf{a}(\nabla u)\cdot n = 0, \text{ on } \partial D, & 0 \le t \le T, \end{cases} \tag{3}$$

where D is the unit square in $\mathbf{R}^2$, $\mathbf{a}(\nabla u) = (a(u_x), a(u_y))$ with $a(s) == s/(1+s^2)$ and n is the outward normal vector on ∂D.

We mainly describe the method for solving Problem 1 and Problem 2 above and briefly explain how we deal with other models.

In the pioneering work of Klaus Höllig [8], infinitely many weak solutions were constructed under the main constitutive assumption that function $a : \mathbf{R} \to \mathbf{R}$ in (1) is piecewise affine, as illustrated in Figure 1 below. The construction of solutions in [8] depends heavily on this constitutive requirement.

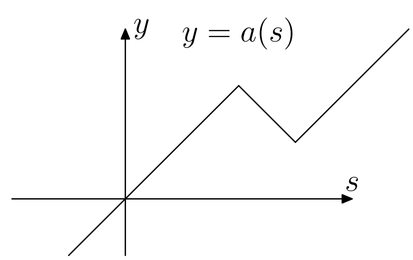

Figure 1: Shape of $y = a(s)$ considered by Höllig

However, it is difficult to generalize Höllig's method to solve (1) with $a(\cdot)$ given by Figure 2 below.

In order to solve (1), we have to look for methods alternative to that of Höllig's.

Problem 1 A generalization of Höllig's model

Hypothesis (A) (illustrated in Fig.2). The function $a : \mathbf{R} \to \mathbf{R}$ satisfies

i) there are two numbers $0 < s_1 < s_2$, such that $a \in C^{2,1}([s_1, s_2]^c) \cap C^0((s_1, s_2)^c)$;.

ii) for $s \in [s_1,\, s_2]^c$, $a'(s) > 0$, $0 < a(s_2) < a(s_1)$;