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THE EXISTENCE AND UNIQUENESS OF A POSITIVE SOLUTION OF AN ELLIPTIC SYSTEM

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Abstract The existence and uniqueness of the positive solution for the generalized Lotka-Volterra competition model for several competing species

$$\Delta u^i + u^i(a - g(u^1, ..., u^N)) = 0 \text{ in } \Omega,$$
$$u^i = 0 \text{ on } \partial\Omega,$$

for $i = 1, ..., N$ were investigated. The techniques used in this paper are elliptic theory, upper-lower solutions, maximum principles and spectrum estimates. The arguments also rely on some detailed properties for the solution of logistic equations.
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1. Introduction

A lot of research has been focused on reaction-diffusion equations modeling of various systems in mathematical biology, especially the elliptic steady states of competitive and predator-prey interacting processes with various boundary conditions. In the earlier literature, investigations into mathematical biology models were concerned with studying those with homogeneous Neumann boundary conditions. From here on, the more important Dirichlet problems, which allow flux across the boundary, became the subject of study.(see [1-8])While necessary and sufficient conditions for the existence of positive solutions to the steady states have been established for rather general types of systems(see [7, 8]), our knowledge about the uniqueness of positive solutions is limited to somewhat rather special systems, whose relative growth rates are linear; the results established are only for the following competition models(see [?], [3-6])

$$\begin{cases} \Delta u^i + u^i(a_i - \beta_i u^i - \sum_{j=1, j\neq i}^N c_{ij} u^j) = 0 \text{ in } \Omega, \\ u^i|_{\partial\Omega} = 0, \\ u^i > 0 \text{ in } \Omega \end{cases}$$

for $i = 1, ..., N$.

The question in this paper concerns the existence and uniqueness of positive coexistence when the relative growth rates are nonlinear, more precisely, the existence and uniqueness of positive steady state of

$$\Delta u^i + u^i \left(a_i - g_i(u^1, ..., u^N)\right) = 0 \text{ in } \Omega,$$
$$u^i = 0 \text{ on } \partial\Omega$$

for $i = 1, ..., N$. Here, a_i's are positive constants, g_i's are C^1 functions, Ω is a bounded domain in R^n and u^i's are densities of N competitive species.

The followings are the problems which we will discuss in this paper.

Problem 1 What are the sufficient conditions for existence and uniqueness of steady state at a fixed reproduction $(a_1, ..., a_N)$ in R^N? When does either one of the species become extinct, i.e. when is either one of the species excluded by the other?

Problem 2 Assume the uniqueness of coexistence state at a fixed reproduction $(a_1, ..., a_N)$, is it possible to do the perturbation to an open ball around $(a_1, ..., a_N)$ with the uniqueness, strictly speaking, is there a neighborhood V of the fixed reproduction rate $(a_1, ..., a_N)$ such that the uniqueness of coexistence state is guaranteed for any reproduction rates $(a'_1, ..., a'_N)$ in V?

Problem 3 This is the generalization of Problem 2. What is the answer of Problem 2 when we have the uniqueness of positive solution to the above equation on the left or right boundary of a closed, convex region Γ of the reproductions $(a_1, ..., a_N)$? Can we still perturb the region Γ to an open set including Γ with the uniqueness?

In Section 3, some sufficient conditions to guarantee the existence, uniqueness of positive solutions are obtained and we also see that they can not coexist for small self-reproduction rates using upper-lower solutions and spectrum estimates, which solves Problem 1. In Sections 4 and 5, we provide the answers for Problems 2 and 3 using elliptic theory, maximum principles and implicit function theorem.

2. Preliminaries

In this section we state some preliminary results which will be useful for our later arguments.

Definition 2.1 (Upper and Lower solutions) *The vector functions* $(\bar{u}^1, ..., \bar{u}^N)$, $(\underline{u}^1, ..., \underline{u}^N)$ *form an upper/lower solution pair for the system*

$$\begin{cases} \Delta u^i + g^i(u^1, ..., u^N) = 0 \text{ in } \Omega \\ u^i = 0 \text{ on } \partial\Omega \end{cases}$$

if for $i = 1, ..., N$

$$\begin{cases} \Delta \bar{u}^i + g^i(u^1, ..., u^{i-1}, \bar{u}^i, u^{i+1}, ..., u^N) \leq 0 \\ \Delta \underline{u}^i + g^i(u^1, ..., u^{i-1}, \underline{u}^i, u^{i+1}, ..., u^N) \geq 0 \\ \text{in } \Omega \text{ for } \underline{u}^j \leq u^j \leq \bar{u}^j, j \neq i, \end{cases}$$