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OBSTACLE PROBLEMS FOR SCALAR GINZBURG-LANDAU EQUATIONS*

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Abstract In this note, we establish some estimates of solutions of the scalar Ginzburg-Landau equation and other nonlinear Laplacian equation $\Delta u = f(x, u)$. This will give an estimate of the Hausdorff dimension for the free boundary of the obstacle problem.
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1. Introduction

Recently there are many interesting results appeared in the study of mathematical theory of super-conductivity. There people considered Dirichlet and Neumann boundary problems. There are a lot of such articles related to bifurcation and stability properties about solutions. People also like to find multiple solutions for complex valued Ginzburg-Landau equations. One interesting problem is the obstacle problem for the scalar Ginzburg-Landau equation. However, the free boundary problems, in particular, obstacle problems, are seldom considered in this theory. Such problem is nature since the Ginzburg-Landau equation has a closed relation with the minimal surface theory. The Obstacle problems for minimal surfaces or for constant mean curvature surfaces have attracted a lot of people. As is well-known, the free boundary problems are very important in science and technology, one may see the article of A. Friedman [1] for more exposition. One such problem for linear elliptic partial differential equations is the obstacle problem, which is considered by many famous mathematicians. In the linear elliptic problem case, L.Caffarelli [2] proved a very beautiful result. In fact, he can show that the solution is $C^{1,1}$ and the free boundary is an n-1 dimensional submanifold. His argument is very delicate. As he pointed out, his method can be used to treat some nonlinear problems. Some of his results and arguments have been extended

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to p-Laplacian problems by K.Lee and H.Shahgholian [3]. One natural question is if a similar result is true for the obstacle problem in the super-conductivity theory.

In this paper, we study the obstacle problem for the scalar Ginzburg-Landau model. Let $D \subset R^n$ be a bounded smooth domain. We are now given a (smooth) bounded function $f(x)$ on ∂D (we assume that f has an extension $f \in C^{2,\mu}(D)$), and a (smooth) function $\phi(x) \in C^{2,\mu}(D)$ with $\varphi(x) < f(x)$ for every $x \in \partial D$. We study the partial differential equation $(GL)_o$:

$$\Delta u + \lambda u(1-u^2) = 0 \quad \text{in } \{u > \varphi\},$$

where $\lambda > 0$ is a (large) constant.

Let $M = |f|_{L^\infty}$. Let $u_0 = inf\{-1, -M\}$ and $u_1 = sup\{1, M\}$. It is clear that u_0 is a sub-solution of $(GL)_o$, and u_1 is a super-solution of $(GL)_o$.

We can get a solution by the direct method. Define K to be the closed convex set

$$K := \{u \in H^1; u_0 \le u \le u_1, u|_{\partial D} = f, u \ge \varphi\}$$

Clearly, since we can extend f on all D such that $f \in K$, K is closed, non-empty convex subset of H^1.

Set

$$J(u) = \int_D |du|^2 + \frac{\lambda}{4}\int_D (u^2-1)^2$$

on K. Then it is easy to see that the infimum is achieved on K. In fact, the minimizer u satisfies the Ginzburg-Landau type equation

$$\Delta u + \lambda u(1-u^2) = 0 \quad \text{in } \{u > \varphi\},$$

where $\lambda > 0$ is a (large) constant. By using a simple comparison argument, it is easy to see that the solution is unique. Let

$$\Omega := \{x; u(x) > \varphi(x)\}$$

Then we meet the question about the regularity of the solution. It is easy to see that since this u is in L^∞, it is $C^{1,\alpha}$, $\alpha \in (0,1)$, (by Theorem 1 in [4]). Furthermore, by adapting the argument in [2], we can show that u is in $C^{1,1}$, and smooth away from the free boundary $S := \{u = \varphi\}$ (see the next section). So the key part is to understand the regularity about the free boundary.

To understand the regularity of this minimizer near the free boundary, without loss of generality, we need only to study the following model problem. In the unit ball of R^n we consider a given function w with the following properties:

(a) $w \ge 0$, $w \in C^{1,1}$;

(b) $\Delta w = g(x)$ in the set $\Omega = \{w > 0\}$;