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HIGHER DEGREE IMMERSED FINITE ELEMENT METHODS FOR SECOND-ORDER ELLIPTIC INTERFACE PROBLEMS

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Abstract. We present higher degree immersed finite element (IFE) spaces that can be used to solve two dimensional second order elliptic interface problems without requiring the mesh to be aligned with the material interfaces. The interpolation errors in the proposed piecewise p^{th} degree spaces yield optimal $\mathcal{O}(h^{p+1})$ and $\mathcal{O}(h^{p})$ convergence rates in the L^2 and broken H^1 norms, respectively, under mesh refinement. A partially penalized method is developed which also converges optimally with the proposed higher degree IFE spaces. While this penalty is not needed when either linear or bilinear IFE space is used, a numerical example is presented to show that it is necessary when a higher degree IFE space is used.

Key words. Immersed finite element, immersed interface, interface problems, Cartesian mesh method, structured mesh, higher degree finite element

1. Introduction

Mathematical modeling of a physical phenomenon in a domain consisting of multiple materials often leads to an interface problem whose exact solution is required to satisfy jump conditions across the material interfaces in addition to the pertinent partial differential equation and the related boundary conditions. Conventional finite element methods with body-fitted meshes can be used to solve interface problems with standard problem independent finite element basis functions. In general, to achieve the optimal convergence of conventional finite element solutions, elements which are cut by the interface should be avoided [6, 9, 12]. This restriction leads to several drawbacks, among which are

(i) The need for remeshing, sometimes many times, when solving problems with moving interfaces. The same difficulty occurs for random interfaces where many problems are solved with different interfaces (from different values of the parameters) to estimate quantities of interest such as the expected solution.
(ii) Excessive mesh refinement to resolve small structures such as thin layers in the domain.
(iii) Prohibition of the use of uniform meshes when solving problems whose interfaces have nontrivial geometries.

In the 1970s and 1980s, Babuška et al. [4, 5] developed the generalized finite element method using the idea of constructing the basis functions on an element by locally solving the interface problem in that element. Instead of generic polynomials, they developed problem dependent local basis functions which may be non-polynomials and are capable of capturing important features of the exact solution. The recently developed IFE methods [2, 3, 11, 14, 15, 18, 19, 20, 21, 22]

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extended this idea and used the jump conditions in the interface problem to construct local basis functions with piecewise polynomials. This idea is similar to Hsieh-Clough-Tocher macro elements [8, 13] in which piecewise cubic polynomials on three sub-triangles are used to satisfy the required continuity. IFE methods use meshes that can be independent of interface geometry; hence they can circumvent the limitations mentioned above for conventional finite element methods.

We note that almost all IFE spaces proposed up to now are based on linear, bilinear, or trilinear polynomials [20, 21, 22, 28] except for those constructed for one dimensional interface problems [2, 3, 11]. It is of great interests to develop higher degree IFE spaces to be used in more efficient schemes such as those based on discontinuous Galerkin formulations with local h and p refinement capabilities. In this manuscript, we present procedures to construct arbitrarily higher degree IFE spaces for solving the typical second order elliptic interface problem on a triangular Cartesian mesh.

Following the usual IFE framework, we use standard finite element shape functions on non-interface elements and we focus on how to construct higher degree IFE shape functions in interface elements with piecewise polynomials that satisfy the interface jump conditions required by an interface problem. However, as first observed in [11], extra constraints need to be carefully introduced in order to uniquely determine higher degree IFE functions with the optimal approximation capability according to the degree of polynomials employed. Based on the idea in [2, 3], we propose to construct higher degree IFE spaces with the interface jump conditions required by the second order elliptic interface problem plus one of the two classes of extended interface jump conditions. Both classes of extended jump conditions involve higher order derivatives. In the first class, the second-order elliptic operator and its normal derivatives of a higher degree IFE function are required to be continuous across the interface. The second class of extended jump conditions enforce the continuity of higher order normal derivatives of the flux of an IFE function.

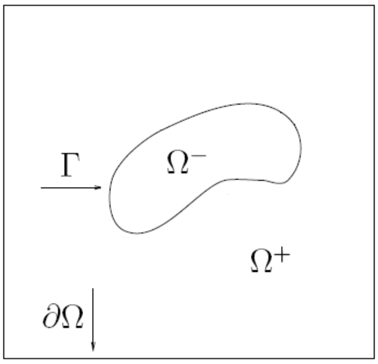

FIGURE 1.1. A two-material domain Ω

IFE functions are generally not continuous across interface edges; hence IFE methods for interface problems are usually nonconforming in the sense that these IFE spaces are not subspaces of $H^1(\Omega)$ to which the exact solution belongs. While a simple Galerkin formulation works satisfactorily for both the linear and bilinear IFE spaces [15, 18, 20, 22], we have noticed that the discontinuity of higher degree IFE functions across interface edges cannot be neglected in developing numerical schemes for solving interface problems. We have found that penalization