J. Partial Diff. Eqs. **16**(2003), 148–156
©International Academic Publishers Vol.16 No.2

THE COMPACTNESS THEOREM OF $SBV_H(\Omega)$ IN THE HEISENBERG GROUP $\mathbf{H}^n$*

Song Yingqing
(School of Science, Nanjing University of Science and Technology, Nanjing 210094;
Hunan City University, Yiyang 413000, China)
(E-mail: songyinqing@263.net)
Yang Xiaoping
(School of Science, Nanjing University of Science and Technology, Nanjing 210094, China)
(E-mail: yangxp@mail.njust.edu.cn)
(Received Apr. 24, 2002; revised Sep. 16, 2002)

Abstract In this paper we aim to show a compactness theorem for $SBV_H(\Omega)$ of special functions u with bounded variation and with $\nabla_H^c u = 0$ in the Heisenberg group $\mathbf{H}^n$.

Key Words SBV_H function; Heisenberg group; decomposition of Radon measure; compactness theorem.

2000 MR Subject Classification 26A45, 49J45.

Chinese Library Classification O175.29, O176.3.

1. Introduction

A result concerning the decomposition of the Radon measure $\nabla_H u$ for $u \in BV_H(\Omega)$ with an open set $\Omega \subset \mathbf{H}^n$ has been obtained in [1] that $\nabla_H u$ can be split into the absolutely continuous part $\nabla_H^a u$, jump part $\nabla_H^j u$, and Cantor part $\nabla_H^c u$, i.e.,

$$\begin{aligned} \nabla_H u &= \nabla_H^a u + \nabla_H^j u + \nabla_H^c u && (1)\\ &= L_u \cdot \mathcal{L}^{2n+1} + \frac{2\omega_{2n-1}}{\omega_{2n+1}}(u^+ - u^-)\nu_u S_d^{Q-1} \lfloor J_u + \nabla_H^c u, && (2)\end{aligned}$$

where $L_u = (L_1, \cdots, L_{2n}) : \Omega \to R^{2n}$ is the approximate Pansu's differential of u, while u^+, u^-, ν_u are respectively the approximate upper limit, lower limit and jump direction of u at a jump point. The three parts on the right-hand side of (2) are mutually singular. The space $SBV_H(\Omega)$ consisted of $u \in BV_H(\Omega)$ with $\nabla_H^c u = 0$ is one of the most suitable frameworks in which lots of problems of calculus of variation can be

*This work is supported by NSF.Grant No.19771048

solved. For instance, a typical variational problem containing a volume and a surface energy is

$$\min\left\{\int_\Omega[|L_u|^2+\alpha(u-g)^2]dh+\beta S_d^{Q-1}(S_u):u\in SBV_H(\Omega)\right\}, \tag{3}$$

where $\alpha,\beta>0, g\in L^\infty(\Omega)$ are fixed, S_d^{Q-1} denotes $(Q-1)$ dimensional spherical Hausdorff measure in $\mathbf{H}^n$ in the sense of the metric d, K runs over all the subsets of $\mathbf{H}^n$ and u varies in $C_H^1(\Omega\setminus K)$. When such a problem is considered by means of the direct method of calculus variation, the compactness theorem of $SBV_H(\Omega)$ will play an important role. Motivated by an idea of [2], one can consider the behavior of $u\in BV_H(\Omega)$ composed with a C_0^1 function to characterize SBV_H functions hence to prove the compactness theorem. In [3] we have investigated the composed function $v=f\circ u$ where $u\in BV_H(\Omega)$ and $f:R^1\to R^1$ is a Lipschitz function and found that the diffuse part (see Definition 2.1), and the jump part of the derivative $\nabla_H v$ behave in a quite different way. In analogy with the classical chain rule formula, $\widetilde{\nabla}_H v=f'(\tilde u)\widetilde{\nabla}_H u$, while $\nabla_H^j v=\dfrac{f(u^+)-f(u^-)}{u^+-u^-}\nabla_H^j u$. Starting from this point, we will establish a useful criterion for membership to $SBV_H(\Omega)$ which can directly be used to prove the compactness of $SBV_H(\Omega)$.

2. Preliminaries

Now we briefly introduce the Heisenberg group $\mathbf{H}^n$ which is generated by the vector fields $X_1,\cdots,X_n,Y_1,\cdots,Y_n$, where $X_j=\frac{\partial}{\partial x_j}+2y_j\frac{\partial}{\partial t}, Y_j=\frac{\partial}{\partial y_j}-2x_j\frac{\partial}{\partial t}, j=1,\cdots,n$ ([4-8]). If $P=[z,t],Q=[\xi,\tau]$ with $z,\xi\in C^n,t,\tau\in R^1$ are points of $\mathbf{H}^n$, we define

$P\cdot Q:=[z+\xi,t+\tau+2Im(z\bar\xi)]$	as the group operation,
$P^{-1}:=[-z,-t]$	as the inverse of P,
$\delta_r(P):=[rz,r^2t]$	as a family of nonisotropic dilations $(r>0)$,
$\tau_P(Q):=P\cdot Q\quad(P \text{ fixed})$	as the group translation from $\mathbf{H}^n$ to $\mathbf{H}^n$,
$\|P\|_\infty:=\max\{|z|,|t|^{\frac12}\}$	as a homogeneous norm of $\mathbf{H}^n$,
$d(P,Q):=\|P^{-1}\cdot Q\|_\infty$	as the distance between points P and Q,
$\pi_{P_0}([z,t]):=\sum\limits_{j=1}^n(x_jX_j(P_0)+y_jY_J(P_0))$	if $P_0\in\mathbf{H}^n, z=x+iy$.

The distance defined as above is equivalent to the C-C distance $d_C(\cdot,\cdot)$ associated with $X_1,\cdots,X_n,Y_1,\cdots,Y_n$ ([9]). $B(P,r),B_r$ means, respectively, closed ball with center P and center 0 and with a common radius r with respect to the metric d.

It is well known that the Hausdorff dimension of $(\mathbf{H}^n,d)$ is $Q=2n+2$. A natural measure dh on $\mathbf{H}^n$ given by the Lebesgue measure $d\mathcal{L}^{2n+1}=dzdt$ on $C^n\times R^1$ is left (right) invariant and is the Haar measure of $\mathbf{H}^n$ ([10]). Throughout this paper $H_d^s(S_d^s)$ denotes the d-metric s-dimensional Hausdorff (spherical Hausdorff) measure ([6,10]),