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GENERALIZED QUASILINEARIZATION METHOD FOR A CLASS OF SEMILINEAR ELLIPTIC SYSTEMS*

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Abstract In this paper, the method of generalized quasilinearization is extended to a class of semilinear elliptic systems, and the sequences which are the solutions of linear differential equations that converge to the unique solution of the given semilinear elliptic system are obtained.
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1. Introduction

The method of quasilinearization was disscussed by Bellman [1], Lakshmikantham and Leela [2] and Lakshmikantham and Vatsala [3,4]. In this paper, it is extended to a class of semilinear elliptic systems.

Consider the following semilinear elliptic system

$$L_i u_i = f_i(x, U), \quad x \in \Omega, \tag{1}$$

$$B_i u_i = \varphi_i(x), \quad x \in \partial\Omega, \tag{2}$$

where $U \equiv (u_1, \cdots, u_n)$, $\Omega \subset R^N$ is a bounded domain with the boundary $\partial\Omega$, and L_i and B_i are elliptic and boundary operators given, respectively, by

$$L_i u_i \equiv -\sum_{j,k=1}^{N} a_{jk}^{(i)}(x) \frac{\partial^2 u_i}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_j^{(i)}(x) \frac{\partial u_i}{\partial x_j} + c^{(i)} u_i,$$

$$B_i u_i \equiv \alpha_i(x) \frac{\partial u_i}{\partial \nu} + \beta_i(x) u_i,$$

where ν is the unit outer normal vector on $\partial\Omega$, and $\alpha_i(x)$, $\beta_i(x) \in C^{1,\alpha}[\partial\Omega]$, $\beta_i(x) > 0$ and $\partial\Omega$ belongs to the $C^{2,\alpha}$. Moreover, it is assumed that for each $i = 1, \cdots, n$, L_i is uniformly elliptic in Ω and $a_{jk}^{(i)}$, $b_j^{(i)}$, $c^{(i)} \in C^\alpha[\bar{\Omega}]$, $c^{(i)}(x) \geq 0$, $\varphi_i \in C^{1,\alpha}(\bar{\Omega})$, $f_i \in C^\alpha(\bar{\Omega} \times R^n)$ in Ω.

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2. Comparison Lemmas

We first give the following comparison result.

Lemma 1 *Let $W, V \in C^2(\bar{\Omega})$ be lower and upper solutions of (1)-(2), that is, W, V satisfy*

$$L_i w_i \leq f_i(x, Z) \text{ in } \Omega, \quad B_i w_i \leq \varphi_i(x) \text{ on } \partial\Omega, \ \text{for } Z \in <W, V> \ [5] \text{ with } z_i = w_i,$$
$$L_i v_i \geq f_i(x, Z) \text{ in } \Omega, \quad B_i v_i \geq \varphi_i(x) \text{ on } \partial\Omega, \ \text{for } Z \in <W, V> \ \text{with } z_i = v_i.$$

Suppose further that

$$|f_i(x, \tilde{U}) - f_i(x, \tilde{V})| \leq K_i |\tilde{U} - \tilde{V}|, \quad \tilde{U}, \tilde{V} \in <W, V>,$$

where $|\tilde{U} - \tilde{V}| = |\tilde{u}_1 - \tilde{v}_1| + \cdots + |\tilde{u}_n - \tilde{v}_n|, \quad c^{(i)}(x) > K_i \geq 0.$

Then $W \leq V$ *[5], namely,* $w_i \leq v_i$, $i = 1, \cdots, n$, $x \in \bar{\Omega}$.

Proof Let $Y(x) = W(x) - V(x)$, namely, $y_i(x) = w_i(x) - v_i(x)$, $i = 1, \cdots, n$. If $y_i(x) \leq 0$ is not true in Ω, then there exists an $\varepsilon > 0$ and $x_0 \in \bar{\Omega}$ such that

$$w_i(x_0) = v_i(x_0) + \varepsilon, \quad w_i(x) \leq v_i(x) + \varepsilon, \quad x \in \bar{\Omega}.$$

If $x_0 \in \partial\Omega$, then $\dfrac{\partial w_i(x_0)}{\partial \nu} \geq \dfrac{\partial v_i(x_0)}{\partial \nu}$ and hence we can get

$$\begin{aligned} B w_i(x_0) &= \alpha_i(x_0) \frac{\partial w_i(x_0)}{\partial \nu} + \beta_i(x_0) w_i(x_0) \\ &\geq \alpha_i(x_0) \frac{\partial v_i(x_0)}{\partial \nu} + \beta_i(x_0)[v_i(x_0) + \varepsilon] > B v_i(x_0), \end{aligned}$$

which is a contradiction.

If $x_0 \in \Omega$, then $\dfrac{\partial w_i(x_0)}{\partial x_j} = \dfrac{\partial v_i(x_0)}{\partial x_j}$, $\sum\limits_{j,k=1}^{N} \left(\dfrac{\partial^2 w_i(x_0)}{\partial x_j \partial x_k} - \dfrac{\partial^2 v_i(x_0)}{\partial x_j \partial x_k} \right) \lambda_j \lambda_k \leq 0$, where λ_j, λ_k are positive constants. Then by using the assumptions it follows that

$$\begin{aligned} f_i(x_0, W(x_0)) &\geq L_i w_i(x_0) \geq L_i \left[v_i(x_0) + \varepsilon \right] \geq f_i(x_0, V(x_0)) + c^{(i)}(x_0) \varepsilon \\ &\geq f_i(x_0, W(x_0)) + \left[c^{(i)}(x_0) - K_i \right] \varepsilon, \end{aligned}$$

which contracts with $c^{(i)}(x) > K_i$. Hence the claim is true and the proof is complete.

Evidently one has the following corollary to Lemma 1.

Corollary 2 *For any* $P = (p_1, \cdots, p_n)$ *with* $p_i \in C^2(\Omega)$ *satisfying*

$$\begin{aligned} & L_i^{(c_0)} p_i \equiv - \sum_{j,k=1}^{N} a_{jk}^{(i)}(x) \frac{\partial^2 p_i}{\partial x_j \partial x_k} + \sum_{j=1}^{N} b_j^{(i)}(x) \frac{\partial p_i}{\partial x_j} + c_0^{(i)} p_i \leq 0, \quad x \in \Omega, \\ & B_i p_i \leq 0, \quad x \in \partial\Omega, \end{aligned} \tag{3}$$

where $c_0^{(i)}(x) > 0$. *Then one has* $p_i(x) \leq 0$ *in* $\bar{\Omega}$.