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GLOBAL EXISTENCE OF CLASSICAL SOLUTION WITH SMALL INITIAL TOTAL VARIATION FOR QUASILINEAR LINEARLY DEGENERATE HYPERBOLIC SYSTEMS

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Abstract In this paper, the author proves the global existence of classical solution to the Cauchy problem with slowly decaying initial data with small initial total variation for general first order quasilinear linearly degenerate hyperbolic systems. This generalizes the corresponding result of A.Bressan for initial data with compact support.

Key Words Linear degeneracy; small initial total variation; slowly decaying initial data; global classical solution; quasilinear hyperbolic system.

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1. Introduction

Consider the following first order quasilinear strictly hyperbolic system

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0, \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^T$ is the unknown vector function of (t, x) and $A(u) = (a_{ij}(u))$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}(u)(i, j = 1, \cdots, n)$.

By strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u). \tag{1.2}$$

Let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))(resp.\ r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^T)$ be a left (resp. right) eigenvector corresponding to $\lambda_i(u)(i = 1, \cdots, n)$:

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \quad (resp.\ A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.3}$$

We have

$$det|l_{ij}(u)| \neq 0 \quad (equivalently,\ det|r_{ij}(u)| \neq 0). \tag{1.4}$$

All $\lambda_i(u), l_{ij}(u)$ and $r_{ij}(u)(i, j = 1, \cdots, n)$ have the same regularity as $a_{ij}(u)(i, j = 1, \cdots, n)$.

Without loss of generality, we may suppose that

$$l_i(u)r_j(u) \equiv \delta_{ij} \quad (i, j = 1 \cdots, n) \tag{1.5}$$

and

$$r_i^T(u)r_i(u) \equiv 1 \quad (i = 1 \cdots, n), \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

Consider the Cauchy problem for the system (1.1) with the following initial data

$$t = 0 : u = \varphi(x), \tag{1.7}$$

where $\varphi(x)$ is a "small" C^1 vector function of x. In the case that $\varphi(x) \in C^2$ possesses compact support and small total variation, suppose that the system (1.1) is strictly hyperbolic and linearly degenerate in the sense of P.D.Lax, A.Bressan proved in [1] the global existence of classical solution to Cauchy problem (1.1) and (1.7). On the other hand, when $\varphi(x) \in C^1$ possesses certain decay properties as $|x| \to +\infty$ and the strictly hyperbolic system (1.1) is only weakly linearly degenerate, Li Ta-tsien, Zhou Yi and Kong De-xing presented in [2–4] a complete result on the global existence of C^1 solution to Cauchy problem (1.1) and (1.7). Moreover, Kong De-xing constructed a counter-example in [5] which shows the necessary decay property of initial data is essential for guaranteeing the global existence of classical solution to Cauchy problem (1.1) and (1.7).

In this paper we will prove that in the case that $\varphi(x) \in C^1$ possesses slowly decaying properties and small total variation, for the strictly hyperbolic and linearly degenerate system (1.1), there exists a unique global classical solution to Cauchy problem (1.1) and (1.7) for all $t \in \mathbb{R}$.

The main result of this paper is the following

Theorem 1.1 *Suppose that in a neighbourhood of $u = 0$, $A(u) \in C^2$ and the system (1.1) is strictly hyperbolic and linearly degenerate i.e. on the domain under consideration*

$$\nabla\lambda_i(u)r_i(u) \equiv 0 \quad (i = 1, \cdots, n). \tag{1.8}$$

Suppose furthermore that the initial data satisfy the following properties:

(i) $\varphi(x) \in C^1$;

(ii) $\varphi(x)$ *satisfies the following slowly decaying property*

$$\bar{\theta} \stackrel{\triangle}{=} \sup_{x\in\mathbb{R}}\{(1 + |x|)(|\varphi(x)| + |\varphi'(x)|)\} < +\infty; \tag{1.9}$$

(iii) *The initial total variation is small enough, namely,*

$$\theta \stackrel{\triangle}{=} \int_{-\infty}^{+\infty} |\varphi'(x)|dx << 1. \tag{1.10}$$