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EXISTENCE AND UNIQUENESS OF PERIODIC SOLUTION OF DELAYED LOGISTIC EQUATION AND ITS ASYMPTOTIC BEHAVIOR*

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Abstract In this paper, our main aim is to study the existence and uniqueness of the periodic solution of delayed Logistic equation and its asymptotic behavior. In case the coefficients are periodic, we give some sufficient conditions for the existence and uniqueness of periodic solution. Furthermore, we also study the effect of time-delay on the solution.
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1. Introduction

The delayed Logistic differential equation

$$\begin{aligned} Lu(t,x) = u(t,x)\left[a(t,x) - b(t,x)u(t,x)\right. \\ \left. -c(t,x)u(t-\tau,x)\right], \quad (t,x) \in R^+ \times \Omega, \end{aligned} \tag{1.1}$$

$$B[u](t,x) = 0, \qquad (t,x) \in R^+ \times \partial\Omega, \tag{1.2}$$

$$u(t,x) = \phi(t,x), \qquad (t,x) \in [-\tau,0] \times \overline{\Omega}, \tag{1.3}$$

is given as a model of single-species population growth. In [1] the authors have studied the case that the coefficients vary periodically and the time delay is given as $\tau = mT$, where m is a positive integer and T is the period. In [2] the coefficients only associated with x has been studied, which implies the steady-state solution is globally asymptotic stable for every given $\tau > 0$. In [3-5] the systems of parabolic equations with delays are studied, which imply the quasi-solutions for single equation may be obtained. In this

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paper we study the existence and uniqueness of the periodic solution of the problem (1.1)(1.2) and the asymptotic behavior of the problem (1.1)-(1.3).

We give the hypotheses below:

(H_1) Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, and L is an operator defined as $L = \partial/\partial t - \triangle$, where $\triangle$ denotes the Laplace operator. The boundary condition is given by

$$B[u] = u \qquad \text{or} \qquad B[u] = \frac{\partial u}{\partial n} + \gamma(x)u.$$

$\gamma(x) \in C^{1+\alpha}(\partial\Omega)$ and $\gamma(x) \geq 0$ on $\partial\Omega$, and $\partial/\partial n$ denotes the outward normal derivative on $\partial\Omega$, $R^+ = (0, \infty)$.

(H_2) The coefficients $a(t,x)$, $b(t,x)$ and $c(t,x)$ are T-periodic in t and Hölder continuous on $[0,T] \times \overline{\Omega}$ with $a(t,x) > 0$; $b(t,x) > 0$; $c(t,x) \geq 0$. We denote a_1, b_1, c_1 and a_2, b_2, c_2 to be the minimum and maximum values of a, b, c on $[0,T] \times \overline{\Omega}$ with $c_2 > 0$ respectively.

(H_3) The time delay τ is a positive constant. $\phi \in C^{0,1}([-\tau, 0] \times \overline{\Omega})$ is a nonnegative bounded function which satisfies the compatibility condition, i.e. $B[\phi(0,x)] = 0$.

Denote $C^{1,\,2}(R^+ \times \Omega)$ to be the set of functions which are once continuously differentiable in $t \in R^+$ and twice continuously differentiable in $x \in \Omega$. Similar notations are used for other function spaces and other domains.

Lemma 1.1 *If there exists a pair of smooth functions $\overline{u}$, $\underline{u} \in C^{1,\,2}(R^+ \times \Omega) \bigcap C([-\tau, \infty) \times \overline{\Omega})$ (called coupled upper and lower solutions) such that $\overline{u} \geq \underline{u}$ on $[-\tau, \infty) \times \overline{\Omega}$, and they satisfy the following inequalities*

$$\begin{aligned} &L\overline{u}(t,x) \geq \overline{u}(t,x)[a(t,x) - b(t,x)\overline{u}(t,x) - c(t,x)\underline{u}(t-\tau,x)], \\ &L\underline{u}(t,x) \leq \underline{u}(t,x)[a(t,x) - b(t,x)\underline{u}(t,x) - c(t,x)\overline{u}(t-\tau,x)], \\ &\qquad\qquad\qquad\qquad\qquad (t,x) \in R^+ \times \Omega, \end{aligned} \tag{1.4}$$

$$B[\overline{u}](t,x) \geq 0 \geq B[\underline{u}](t,x), \qquad (t,x) \in R^+ \times \partial\Omega, \tag{1.5}$$

$$\overline{u}(t,x) \geq \phi(t,x) \geq \underline{u}(t,x), \qquad (t,x) \in [-\tau, 0] \times \overline{\Omega}, \tag{1.6}$$

then the initial-boundary value problem (1.1)-(1.3) has a unique solution $u \in C^{1,\,2}(R^+ \times \Omega) \bigcap C([-\tau,\ \infty) \times \overline{\Omega})$ with $\overline{u} \geq u \geq \underline{u}$ on $[-\tau, \infty) \times \overline{\Omega}$.

In the case $\overline{u}$, $\underline{u}$ satisfy (1.4)(1.5) with $\overline{u} \geq \underline{u}$ on $R^+ \times \overline{\Omega}$, we also call $\overline{u}$, $\underline{u}$ a pair of upper and lower solutions of the problem (1.1)(1.2). For the proof of *Lemma 1.1* we can refer to [6, 7]. As there always exists a positive number α large enough such that $\phi(t,x) \leq \alpha$ on $[-\tau, 0] \times \overline{\Omega}$, it is easy to check that α and 0 is a pair of upper and lower solutions of the problem (1.1)–(1.3), then from *Lemma 1.1* we can get a unique