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ON THE $W^{1,q}$ ESTIMATE FOR WEAK SOLUTIONS TO A CLASS OF DIVERGENCE ELLIPTIC EQUATIONS*

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Abstract Local $W^{1,q}$ estimates for weak solutions to a class of equations in divergence form

$$D_i(a_{ij}(x)|Du|^{p-2}D_ju) = 0$$

are obtained, where $q > p$ is given. These estimates are very important in obtaining higher regularity for the weak solutions to elliptic equations.

Key Words Divergence elliptic equation; local $W^{1,q}$ estimate; reverse Hölder inequality.

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1. Introduction

Using compactness method, Avellanda and Lin Fanghua in [1] obtained L^p theory for elliptic systems of periodic structure

$$L^\varepsilon = -\frac{\partial}{\partial x^\alpha}\left[A_{ij}^{\alpha\beta}(\frac{x}{\varepsilon})\frac{\partial}{\partial x^\beta}\right] = f.$$

Using the results in [1], they in [2] also obtained $C^{0,\alpha}, C^{1,\alpha}$ and $C^{0,1}$ regularity for homogenization problem:

$$\begin{cases} \displaystyle\sum_{i,j=1}^n a^{ij}(\frac{x}{\varepsilon})\frac{\partial^2 u_\varepsilon}{\partial x^i x^j} = f(x), & x \in D, \\ u_\varepsilon(x) = g(x), & x \in \partial D, \end{cases}$$

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under certain conditions, where $\varepsilon > 0, D$ is smooth domain in R^n. Using Calderón-Zygmund decompositions theorem [3] and measure theory [4], Caffarelli and Petal in [5] established a determinant theorem for the weak solutions which have higher integrability to a class of homogenization problems, and using this theorem, the authors obtained higher integrability for weak solutions to equations

$$\mathrm{div}(a(x, Du)) = 0, \tag{1}$$

then using this result, the authors obtained corresponding results for homogenization problem with periodic structure in [1] and [2]. By the method different from that in [1-2] and [5], Kilpeläinen and Koskela [6] obtained global integrability for the weak solutions to the equation (1). Li Gongbao and Martio [7] obtained local and global integrability for the gradient of the weak solutions to the equation (1). They also in [8] obtained that the weak solution to the equation (1) with very weak boundary value is exclusive. The L^p estimates established in [1] played crucial role in obtaining the results in [2]. But Caffarelli and Petal in [5] didn't obtain corresponding L^p estimates.

In this paper, we discuss the weak solutions in $W^{1,p}$ to the following equation

$$D_i(a_{ij}(x)|Du|^{p-2}D_j u) = 0. \tag{2}$$

Using the method in [5], we obtain L^q integrability for the gradient of the weak solutions to the equation (2),where q is given to be bigger than p, then establish the reverse Hölder inequality for the equation (2) by the method in [9] and [10], and obtain local $W^{1,q}$ estimate for weak solutions to the equation (2).

2. $W^{1,q}$ Estimate

In this section, we discuss the weak solution in $W^{1,p}$ to the elliptic equation of divergence structure

$$D_i(a_{ij}(x)|Du|^{p-2}D_j u) = 0, \tag{3}$$

where, a_{ij} satisfies:

$$\lambda|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \tag{4}$$

where, $\lambda, \Lambda > 0$ are constants.

We have the following theorem and corollary:

Theorem 2.1 *Suppose q is bigger than p; if there exists $\epsilon > 0$,*

$$||a(x) - I|| \leq \epsilon, \tag{5}$$

where $a(x) = (a_{ij}), I$ is identical matrix and if $u \in W^{1,p}$ is a weak solution to the equation (3), then $W^{1,q}_{loc}(\Omega)$, and for $\forall R, B_R \subset \Omega$,

$$\left[\fint_{B_{\frac{R}{2}}} (|Du|^q + |u|^q)dx\right]^{\frac{1}{q}} \leq \left[\fint_{B_R} (|Du|^p + |u|^p)dx\right]^{\frac{1}{p}}, \tag{6}$$