J. Partial Diff. Eqs. **15**(2002), 61–67
©International Academic Publishers Vol.15 No.3

BLOW-UP OF SOLUTIONS OF A QUASILINEAR WAVE EQUATION WITH NONLINEAR DISSIPATION

Abbes Benaissa[1] and Salim. A. Messaoudi[2]
([1] Department of Mathematics,
Djillali Liabes University,
P. O. Box 89, Sidi Bel Abbes 22000, ALGERIA.)
([2] Mathematical Sciences Department,
KFUPM, Dhahran 31261, SAUDI ARABIA
E-mail: messaoud@kfupm.edu.sa)
(Received Jan. 8, 2002; revised Apr. 24, 2002)

Abstract In this paper we establish the blow up of solutions to the quasilinear wave equation with a nonlinear dissipative term

$$u_{tt} - M(\|A^{1/2}u\|_2^2)Au + |u_t|^\beta u_t = |u|^p u \quad x \in \Omega,\, t > 0$$

Key Words Blowup; Quasilinear wave equation; Finite time.
2000 MR Subject Classification 35L35, 35L70.
Chinese Library Classification O175.25

1. Introduction

We consider the initial boundary value problem(IBVP) for the nonlinear wave equation

$$\begin{cases} u_{tt} - M(\|A^{1/2}u\|_2^2)Au + |u_t|^\beta u_t = |u|^p u, & x \in \Omega,\, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\, t \geq 0, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{P}$$

where

$$Au = e^{-\Phi(x)}\mathrm{div}(e^{\Phi(x)}\nabla u), \quad ||A^{1/2}u||^2 = \int_\Omega e^{\Phi(x)}|\nabla u|^2\,dx,$$

$p > 0$ is a constant, β is a positive constant, $M : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function, Φ is a bounded function, and $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary Γ so that the divergence theorem can be applied.

When $M \equiv 1$ and $\Phi \equiv 0$, for the case of no dissipation (i.e. (P) without the term $|u_t|^\beta u_t$), it is well known that the source term $|u|^p u$ is responsible for finite blow up (global nonexistence) of solutions with negative initial energy (see [1] and [2]). The interaction between the damping term and the source has been first considered by

Levine [3], [4] for the case with linear dissipation of the form τu_t ($\tau > 0$). In his work, Levine showed that solutions with negative initial energy blow up in finite time. In [5] Georgiev and Todorova extended Levine's result to the case of nonlinear damping of the form $|u_t|^\beta u_t$. This result was generalized to an abstract setup by Levine and Serrin [6], Levine and Park [7] and Vitillaro [8]. In [9] Messaoudi extended the result of Levine to the situation where $\Phi \neq 0$.

When $\Phi \equiv 0$ and M is not a constant function the equation without the damping and source terms is often called the wave equation of Kirchhoff type which has been introduced in order to study the nonlinear vibrations of an elastic string by Kirchhoff [10]. The existence of local and global solutions in Sobolev and Gevrey classes was investigated by many authors (see [11-14]).

In the present paper, we study the behavior of the solutions of an initial-boundary value problem for the nonlinear wave equation. We shall show that, for suitably chosen initial data, any solution, which exists in appropriate Sobolev spaces, blows up in finite time.

We will apply to our problem (P) the argument introduced in [5] to prove the blow-up of solutions. This work extends an earlier one in [15], where the dissipative term was linear.

2. Main Results

In order to state our main result, we make the following hypotheses:

$$M \in C(\mathbb{R}_+, \mathbb{R}_+) \text{ and } m\ \overline{M}(s) \geq s\ M(s), \forall s \geq 0, \tag{1}$$

where $\overline{M}(s) = \int_0^s M(k)\, dk$, $m \geq 1$.

$$p > \max\{\beta, 2m-2\}. \tag{2}$$

Theorem 2.1 *Assume that (1) and (2) hold. Then for any initial data* $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, *satisfying*

$$E(0) = \frac{1}{2}\int_\Omega e^{\Phi(x)} u_1^2\, dx + \frac{1}{2}\overline{M}\left(\int_\Omega e^{\Phi(x)} |\nabla_x u_0|^2\, dx\right) - \frac{1}{p+2}\int_\Omega e^{\Phi(x)} |u_0|^{p+2}\, dx < 0, \tag{3}$$

the solution of (P) *blows up in finite time.*

Proof We set

$$E(t) = \frac{1}{2}\int_\Omega e^{\Phi(x)} u_t^2\, dx + \frac{1}{2}\overline{M}\left(\int_\Omega e^{\Phi(x)} |\nabla_x u|^2\, dx\right) - \frac{1}{p+2}\int_\Omega e^{\Phi(x)} |u|^{p+2}\, dx. \tag{4}$$

By multiplying the equation of (P) by $e^{\Phi(x)} u_t(x,t)$ and integrating over Ω, we get

$$E'(t) = -\int_\Omega e^{\Phi(x)} |u_t|^{\beta+2}(x,t)\, dx. \tag{5}$$