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CONVERGENCE ANALYSIS OF YEE SCHEMES FOR MAXWELL'S EQUATIONS IN DEBYE AND LORENTZ DISPERSIVE MEDIA

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Abstract. We present discrete energy decay results for the Yee scheme applied to Maxwell's equations in Debye and Lorentz dispersive media. These estimates provide stability conditions for the Yee scheme in the corresponding media. In particular, we show that the stability conditions are the same as those for the Yee scheme in a nondispersive dielectric. However, energy decay for the Maxwell-Debye and Maxwell-Lorentz models indicate that the Yee schemes are dissipative. The energy decay results are then used to prove the convergence of the Yee schemes for the dispersive models. We also show that the Yee schemes preserve the Gauss divergence laws on its discrete mesh. Numerical simulations are provided to illustrate the theoretical results.

Key words. Maxwell's equations, Debye, Lorentz dispersive materials, Yee, FDTD method, energy decay, convergence analysis.

1. Introduction

The Yee scheme is a finite difference time domain (FDTD) numerical technique for the discretization of Maxwell's equations in a non-dispersive medium such as free space. It was first presented in [35]. The Yee scheme was extended to discretize Maxwell's equations in linear dispersive media and analyzed in a series of papers [4, 9, 13, 18, 19, 20, 31] involving dispersive media models such as the Debye [14, 20], Lorentz [18, 30], cold plasma [13, 36] and Cole-Cole [9, 12] models among others. Fourier analysis of the Yee scheme in such dispersive media (see for e.g. [4, 31]) indicate that the Yee scheme is stable under the same stability condition as that in a corresponding (having the same relative permittivity) non-dispersive dielectric. However, the Yee scheme in dispersive media is dissipative, unlike its counterpart in a non-dispersive, non-conductive medium, and in addition is more dispersive [5, 32]. The time step in the Yee scheme needs to be chosen to resolve all the time scales associated with a particular dispersive medium such as relaxation times, resonance times, and incident wave periods [32]. Maxwell's equations in such media have been shown to constitute a stiff problem and the time step needed to resolve waves in the numerical grid can be extremely small [32]. Research on the construction and analysis of Yee type finite difference time domain methods for Maxwell's equations in dispersive media is an area of active interest. We refer the reader to the book [33] and the numerous references therein for an introduction to the Yee scheme and its properties.

In this paper we present for the first time an analysis of the Yee scheme in Debye (Maxwell-Debye) and Lorentz (Maxwell-Lorentz) media by deriving energy decay results that indicate the conditional stability and dissipative nature of the schemes. We also present a full convergence analysis of the Yee schemes for the Maxwell-Debye and Maxwell-Lorentz models using the derived energy decay results. Energy methods based on variational techniques for analyzing stability

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and convergence properties of the Yee scheme in a lossy non-dispersive medium and operator splitting FDTD techniques have recently been published in the literature, see for example [7, 10, 15]. Finite element methods (FEM) and discontinuous Galerkin (DG) methods for Maxwell's equations in various dispersive media have also recently been published, for example see [1, 16, 21, 22, 23, 24, 25, 26, 34] and references therein.

We construct exact solutions based on numerical dispersion relations for the Maxwell-Debye and Maxwell-Lorentz models which are useful in understanding the decay of discrete energies in numerical methods for these models. We use these exact solutions to illustrate our stability and convergence analyses in our numerical simulations of the Yee schemes.

The outline of the paper is as follows. In Section 2 we present two dispersive media models and construct the Maxwell-Debye model and Maxwell-Lorentz model in two dimensions. We recall energy decay results for these models from the literature [23]. In Section 3 we outline the discrete meshes and spaces that the electric, magnetic and polarization fields are discretized on and establish discrete curl operators and their properties. In Sections 4 and 5 we recall the Yee schemes for the Maxwell-Debye and the Maxwell-Lorentz models, respectively. For both models we show that the corresponding Yee schemes are second-order accurate in time, establish discrete energy decay results and prove the conditional convergence of the corresponding Yee schemes. In addition, we show that these schemes satisfy the Gauss divergence laws on the discrete Yee mesh. Numerical simulations based on exact solutions are presented in Sections 6 and 7 that illustrate the stability and convergence analyses. Finally, conclusions are made in Section 8.

2. Maxwell's Equations in Dispersive Dielectrics

We consider Maxwell's equations which govern the electric field $\mathbf{E}$ and the magnetic field $\mathbf{H}$ in a domain $\Omega \subset \mathbb{R}^3$ from time 0 to T given as

$$\frac{\partial \mathbf{D}}{\partial t} - \nabla \times \mathbf{H} = \mathbf{0} \text{ in } \Omega \times (0,T), \tag{2.1a}$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = \mathbf{0} \text{ in } \Omega \times (0,T), \tag{2.1b}$$

$$\nabla \cdot \mathbf{D} = 0 = \nabla \cdot \mathbf{B} \text{ in } \Omega \times (0,T), \tag{2.1c}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \text{ on } \partial\Omega \times (0,T), \tag{2.1d}$$

$$\mathbf{E}(0,\mathbf{x}) = \mathbf{E}_0;\ \mathbf{H}(0,\mathbf{x}) = \mathbf{H}_0 \text{ in } \Omega. \tag{2.1e}$$

The fields $\mathbf{D}, \mathbf{B}$ are the electric and magnetic flux densities respectively. On the boundary, $\partial\Omega$, we impose a perfect conducting (PEC) boundary condition (2.1d), where the vector $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. Lastly, we add initial conditions (2.1e) to the system.

Within the dielectric medium we have constitutive relations that relate the flux densities $\mathbf{D}, \mathbf{B}$ to the electric and magnetic fields, respectively, as

$$\mathbf{D} = \epsilon_0 \epsilon_\infty \mathbf{E} + \mathbf{P}, \tag{2.2a}$$

$$\mathbf{B} = \mu_0 \mathbf{H}, \tag{2.2b}$$

where the constants ϵ_0 and μ_0 are the permittivity and permeability of free space, and are connected to the speed of light in vacuum, c_0, by $c_0 = 1/\sqrt{\epsilon_0 \mu_0}$. The