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THE BOUNDARY REGULARITY OF PSEUDO-HOLOMORPHIC DISKS*

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Abstract In this paper we will prove the continuity, the C^k-regularity after deforming suitably the domain, and the Hölder continuity, of the weakly pseudo-holomorphic disk with its boundary in a singular totally-real subvariety with only corners as its singularities.

Key Words Pseudo-holomorphic disk; totally-real submanifold; continuity; Hölder continuity; corner.

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1. Introduction

Since M. Gromov introduced pseudo-holomorphic curves into the symplectic geometry in 1985 [1], the application of pseudo-holomorphic curves to the symplectic geometry has become a main tool in the study of symplectic manifolds and achieved great success. Pseudo-holomorphic disks play a great role in the study of lagrangian submanifolds in a symplectic manifold [2–4].

The regularity of pseudo-holomorphic curves at the interior points was established by several authors [1,5,6] and the boundary regularity was established by Ye[5] with a slightly different setting and by Sikorav[6] who assumed the continuity of those curves in the smooth boundary case. M. Gromov suggested to deal with the regularity and the gradient estimates at the boundary points by making a reflection across the boundary and reducing this problem to the interior point case. The reflection argument indeed was carried out by Pansu[7] under the assumptions which require that the boundary manifold be real analytic and the almost complex structure be integrable near the boundary manifold.

In many applications one needs the corresponding regularity results for the pseudo-holomorphic disks with their boundaries in a totally-real subvariety with corners, for examples, such as in defining Floer homology for the Lagrangian intersections and

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in defining the invariants of Gromov-Witten type [8,9] in above situations, in both cases one needs the compactness of the moduli spaces of pseudo-holomorphic disks, so one has to deal with the bubblings of a family of weakly pseudo-holomorphic disks at the corners. We cannot expect C^1-regularity at the corners because the pseudo-holomorphic maps are conformal, but we can change the domain suitably and obtain the C^k-regularity for any positive integer k and discuss the bubblings at the corners under the C^k topology.

Let (M, J) be a closed oriented smooth almost complex manifold of dimension $2n$ with a smooth almost complex structure J. On M a Riemannian metric $<,>$ is assumed and J is compatible with this metric in the sense that J is an isometry. Let L be a totally-real submanifold which may have corner points as its singular points. Let D^2 denote the open unit disk in the complex plane with the standard complex structure.

First we give a formulation of the weakly pseudo-holomorphic disks with the natural boundary conditions. Let $u \in W^{1,2}(D^2, \partial D^2; M, L)$ be smooth away from the singularities, and $X \in W^{1,2}(D^2, \partial D^2; u^*(TM), u^*(TL)) \cap L^\infty$ be a vector field along u with its boundary values tangent to L at the smooth points. It is easy to see that $< \partial u/\partial \nu, X >= 0$, where ν denotes the inward unit vector normal to ∂D^2 and X a vector tangent to L at the smooth point of L. So by differentiating the Cauchy-Riemann equations, we have

$$\frac{\partial^2 u}{\partial s^2} + \frac{\partial^2 u}{\partial t^2} + \nabla_s J(u)\frac{\partial u}{\partial t} - \nabla_t J(u)\frac{\partial u}{\partial s} = 0$$

Integrating this equality by parts gives ii) in following definition.

Definition 1 *We call map $u \in W^{1,2}(D^2, \partial D^2; M, L)$ a weakly pseudo-holomorphic disk with its boundary in L if*

i)
$$\bar\partial_J u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0 \tag{1}$$

a.e. on D^2;

ii)
$$\iint_{D^2} \langle \frac{\partial u}{\partial s}, \frac{\partial \phi}{\partial s}\rangle + \langle \frac{\partial u}{\partial t}, \frac{\partial \phi}{\partial t}\rangle + \iint_{D^2} \langle \nabla_s J(u)\frac{\partial u}{\partial t}, \phi\rangle - \langle \nabla_t J(u)\frac{\partial u}{\partial s}, \phi\rangle = 0 \tag{2}$$

for any $W^{1,2} \cap L^\infty$-vector field ϕ along u with its boundary values tangential to L a.e.. Here that $u(\partial D^2)$ is included in L means the L^2-trace of u is included in L, where (s,t) denotes the coordinate variable on D^2.

Readers may compare the definition of the weakly pseudo-holomorphic disk in this paper with Ye's definition of weakly pseudo-holomorphic disks, especially the normal boundary conditions in Ye [5].

Now we give the exact meaning of the angles at the corner points.

Definition 2 *Let M be a smooth almost complex manifold with a smooth almost complex structure $J(\cdot)$ and L be a totally-real subvariety with respect to $J(\cdot)$. The angle at $p \in L$ is said to be α if the followings hold:*