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WELL-POSEDNESS, DECAY ESTIMATES AND BLOW-UP THEOREM FOR THE FORCED NLS*

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Abstract In this article we prove that the following NLS $iu_t = u_{xx} - g|u|^{p-1}u, g > 0, x, t > 0$ with either Dirichlet or Robin boundary condition at $x = 0$ is well-posed. L^{p+1} decay estimates, blow-up theorem and numerical results are also given.

Key Words Nonlinear Schrödinger equation; well-posedness; decay estimates; blow-up.

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1. Introduction

Boundary value problems for important evolution equations often are called forced problems. Often these problems have significant physical implications. For example, in ionospheric modification experiments, one directs a radio frequency wave at the ionosphere. At the reflection point of the wave, a sufficient level of electron plasma waves is excited to make the nonlinear behavior important [1,2]. This may be described by the NLS equation with the cubic nonlinear term and a nonlinear boundary condition

$$\begin{cases} iq_t = q_{xx} \pm 2|q|^2 q, & x, t \in \mathbf{R}^+ \\ q(x,0) = h(x), q(0,t) = g(t) \end{cases} \tag{1.1}$$

where $h(x)$ decays for large x and the given functions $h(x), g(t)$ have appropriate smoothness, and satisfy the necessary compatibility conditions. For (1.1), global existence, well-posedness and blow-up result were established when $h \in H^2[0,\infty), Q \in C^2[0,\infty)$[3,4].

In this paper, we study the following NLS with a general nonlinear term $-g|u|^{p-1}u$ for $p > 1, g > 0$:

$$\begin{cases} iu_t = u_{xx} - g|u|^{p-1}u, & x, t \in \mathbf{R}^+ \\ u(x,0) = h(x) \end{cases} \tag{1.2}$$

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with either Dirichlet boundary condition $u(0,t) = Q(t)$ or Robin boundary condition $u_x(0,t) + \alpha u(0,t) = R(t)$, where α is real. Under the assumption that $h \in H^2[0,\infty), Q$ or $R \in C^2[0,\infty)$, there exists a unique global classical solution $u \in C^1([0,\infty), L^2[0,\infty)) \cap C^0([0,\infty), H^2[0,\infty))$[3]. Let $P(t) = u_x(0,t)$, the following three identities can be easily verified (in the case of Robin boundary condition, P is replaced by $R - \alpha Q$):

$$\partial_t \int_0^\infty |u|^2 dx = -2\mathrm{Im}(P\bar{Q}) \tag{1.3}$$

$$\partial_t \int_0^\infty \left(|u_x|^2 + \frac{2g}{p+1}|u|^{p+1}\right) dx = -2\mathrm{Re}P\bar{Q}' \tag{1.4}$$

and

$$\partial_t \int_0^\infty u\bar{u}_x dx = -Q\bar{Q}' + i|P|^2 - i\frac{2g}{p+1}|Q|^{p+1} \tag{1.5}$$

In the following, we prove well-posedness for the above problem with either boundary condition, give L^{p+1} decay estimates *via* a pseudoconformal identity and present blow-up result.

2. Well-Posedness Results

Consider (1.2) for $0 \le t \le T$ and assume that for some $M > 0, \|Q\|_{C^2[0,T]} < M$ or $\|R\|_{C^2[0,T]} < M$, depending on the type of boundary condition. Also we assume that $\|h\|_{H^2(\mathbf{R}^+)} \le M$. For the Dirichlet boundary value problem, assume that u, v solve (1.2) with boundary-initial data (Q, u_0) and (Q_1, v_0) both lying in $C^2[0,T] \times H^2(\mathbf{R}^+) = X$. By global existence theorem, there exists a constant $\tilde{\lambda} > 0$ that only depends on M and T such that $\|u\|_{H^1(\mathbf{R}^+)} \le \tilde{\lambda}$ for $t \in [0,T]$ thus $\|u\|_\infty \le c_0\|u'\|_2^{\frac{1}{2}}\|u\|_2^{\frac{1}{2}} \le \lambda$. Clearly, the map $f : X \to Y = C^1(L^2,[0,T]) \cap C^0(H^2,[0,T]) via (Q, u_0) \mapsto u$ is well-defined. Let $z = (Q, u_0), z_1 = (Q_1, v_0) \in X, \|z\|_X = \max\{\|Q\|_{C^2[0,T]}, \|h\|_{2,2}\} < M, \|z_1\|_X < M$ and

$$w = \Delta u = v - u, \Delta z = z_1 - z = (\Delta Q, w_0) = (Q_1 - Q, v_0 - u_0) \tag{2.1}$$

Since $v = w + u$ satisfies (1.2) as well, one has $i(w_t + u_t) = w_{xx} + u_{xx} - g|w + u|^{p+1}(w+u)$ where w satisfies the following variable-coefficient, initial-value, boundary-value problem:

$$\begin{cases} iw_t = w_{xx} - g|w+u|^{p+1}(w+u) + u_{xx} - iu_t = w_{xx} + G(w,t) \\ w(0,t) = \Delta Q, w_0 = v_0 - u_0 \end{cases} \tag{2.2}$$

Let $\Delta P = P_1 - P = v_x(0,t) - u_x(0,t)$. From (2.2) one has

$$i\partial_t|w|^2 = iw_t\bar{w} + iw\bar{w}_t = 2i\mathrm{Im}(w_{xx}\bar{w} + \bar{w}G(w,t)) \tag{2.3}$$