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VORTEX MOTION LAW OF AN EVOLUTIONARY GINZBURG-LANDAU EQUATION IN 2 DIMENSIONS*

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Abstract We study the asymptotic behavior of solutions to an evolutionary Ginzburg-Landau equation. We also study the dynamical law of Ginzburg-Landau vortices of this equation under the Neuman boundary conditions. Away from the vortices, we use some measure theoretic arguments used by F.H.Lin in [1] to show the strong convergence of solutions. This is a continuation of our earlier work [2].

Key Words Ginzburg-Landau equations; vortex motion; asymptotic behavior; ε-regularity.

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1. Introduction

We consider the following problem:

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon + \frac{1}{\varepsilon^2}(\beta^2(x) - |u_\varepsilon|^2)u_\varepsilon, \quad (x,t) \in \Omega \times R_+ \tag{1.1}$$

$$u_\varepsilon(x,0) = \beta u_\varepsilon^0(x), \quad x \in \Omega \tag{1.2}$$

$$\frac{\partial u_\varepsilon}{\partial \nu}(x,t) = 0, \quad x \in \partial\Omega, \quad t \geq 0 \tag{1.3}$$

where Ω is a smooth bounded domain in R^2, ν the exterior unit normal vector along $\partial\Omega$. $\beta(x) : \Omega \to R$ is a smooth function (say C^3) with positive lower bound. u_ε : $\Omega \times R_+ \to R^2$.

The initial datum $\beta u_\varepsilon^0(x)$ is smooth and satisfies (1.3). In addition, it also satisfies the following assumptions:

$$\|u_\varepsilon^0(x)\|_{C(\bar\Omega)} \leq K \tag{1.4}$$

$$\int_\Omega \rho^2(x)[|\nabla u_\varepsilon^0(x)|^2 + \frac{1}{\varepsilon^2}\beta^2(1 - |u_\varepsilon^0|^2)^2]dx \leq K \tag{1.5}$$

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for a constant K and some m distint points $b_1, \cdots, b_m$ in Ω, where $\rho(x) = \min\{|x - b_j| : j = 1, 2, \cdots, m\}$.

$$\begin{aligned} E(u_\varepsilon^0) &= \frac{1}{2}\int_\Omega [|\nabla u_\varepsilon^0|^2 + \frac{1}{2\varepsilon^2}\beta^2(x)(|u_\varepsilon^0(x)|^2 - 1)^2]dx \\ &\leq K[|\ln\varepsilon| + 1] \end{aligned} \tag{1.6}$$

The system (1.1)–(1.3) can be viewed as a simplified evolutionary Ginzburg-Landau equation in the theory superconductivity of inhomonogence ([3]).

The aim of this article is to understand the dynamics of vortices, or zeros, of solutions u of (1.1)–(1.3). Its importance to the theory of superconductivity and applications is addressed in many earlier work ([3–7]).

Now we briefly describe some mathematical advances concerning this problem. In $\beta = 1$, the dynamical law for vortices was formally derived in [4,8]. The first rigorous mathematical proof of this dynamical law, which is of form $\frac{\partial}{\partial t}a(t) = -\nabla w(a(t))$, was given by F.H.Lin in [5,9]. See also [10, Lecture 3]. In [5,9], one allows the vortices of degree ± 1 and assumes that they have the same sign. For the vortices of degree ± 1 (possibly of different signs), the same dynamical law was shown later in [11]. We refer to [1] for vortex dynamics under the Neumann boundary conditions or pinning conditions. In the 3-dimensional case, $\beta = 1$, a similar dynamical law was also established in [1] for nearly paralled filaments. The short-time dynamical law for codimension 2 interfaces in higher dimensions was shown in [1]. When $\beta \not\equiv 1$, in the 2-dimensional case, the dynamical law was established in [12] under the first boundary condition. But, here one proves only that u_ε conveges weakly to the limit function in H^1_{loc} as $\varepsilon \to 0^+$.

The main goal of the present paper is to examine the vortex dynamics without topological constraints, and proves that u_ε converges strongly to the limit function in H^1_{loc} as $\varepsilon \to 0^+$.

To understand the behavior of u of (1.1)–(1.3) as $t \to \infty$, one has to look at steady state solutions u_ε, that is, the minimizer of the energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left[|\nabla u|^2 + \frac{1}{2\varepsilon^2}(\beta^2 - |u|^2)^2\right] dx$$

A complete characterizaton of asymptotic behavier (as $\varepsilon \to 0^+$) of vortices of u_ε is given in the recent work [2].

Now we claim our main theorem.

Theorem 1.1 *Assume that* $\beta \in C^3(\bar\Omega)$ *and* $\beta_0 = \min_{\bar\Omega} \beta(x) > 0$. *Under the assumptions* (1.4)–(1.6), *one has, for any* $0 \leq t \leq T$, *that*

$$u_\varepsilon(x,t) \to u_*(x,t) \tag{1.7}$$

strongly in $H^1_{\text{loc}}(\bar\Omega \times [0,T] \backslash \{(a_j(t), t) : t \in [0,T], j = 1, 2, \cdot, m\})$. *Here the convergence is understood in the sense that for any sequence of* ε' *going to zero, there is a subsequence*