J. Partial Diff. Eqs. 14(2001), 207–219
©International Academic Publishers Vol.14, No.3

THE ALTERNATING-DIRECTION IMPLICIT CHARACTERISTIC FINITE ELEMENT METHODS FOR THE THREE-DIMENSIONAL GENERALIZED NERVE CONDUCTION EQUATION*

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(Received Oct. 11, 2000; revised Mar. 30, 2001)

Abstract A complete convergence analysis is given for two-level scheme of the alternating-direction implicit characteristic finite element method for the approximate solution of the three-dimensional generalized nerve conduction equation. By use of the calculation of vector product, H^{-1} norm estimates, and a priori estimate theory and technique, the optimal order estimate in L^2 is obtained.
Key Words Nerve conduction equation; the alternating-direction implicit; characteristic finite element; optimal order estimate in L^2.
1991 MR Subject Classification 65M, 65N.
Chinese Library Classification O241.

1. Introduction

In this paper, we are concerned with numerical approximation to the three-dimensional generalized nerve conduction equation

$$u_{tt} + b(x,u,u_t)\cdot\nabla u_t - \Delta u_t = f(u)u_t - g(u),\ \ x\in\Omega,\ \ t\in J=[0,T] \tag{1.1 a}$$

$$\frac{\partial u}{\partial n} = \frac{\partial u_t}{\partial n} = 0,\ \ x\in\partial\Omega,\ \ t\in J \tag{1.1 b}$$

$$u(x,0)=u_0(x),\ \ u_t(x,0)=w_0(x),\ \ x\in\Omega \tag{1.1 c}$$

where $\Omega=[0,1]^3$, $\partial\Omega$ denotes the boundary of Ω, $b(x,u,u_t)=\{b_1(x,u,u_t), b_2(x,u,u_t), b_3(x,u,u_t)\}$. $u_0(x)$ and $w_0(x)$ and known functions in $H_0^r(\Omega)\cap H^{r+1}(\Omega)$.

In the process of nerve conduction, nerve conduction signal u and its variability with respect to time and space can be used to character by the three-dimensional pseudohyperbolic equation[1] in mathematics. Some results about the uniqueness and existence, and the asymptotic behavior of solutions for this class equation can be found in [2–4]. There is no work from the numerical analysis viewpoint to our knowledge.

* The project supported by National Natural Science Foundation of China and the Doctoral Foundation of the State Educational Ministry of China.

Since pseudohyperbolic equation is a class of new nonlinear development equation and has deepen physical background in [5], it is neccessary to develop the studies all-sidely and deeply either from the theoritical point of view or from the numerical analysis and practical point of view.

The alternating-direction implicit finite element method is not only used to reduce a multi-dimensional complex problem to the solution of collection of independent one-dimensional problems and economize the storage capacity of computer, but also to write a very efficient code for solving large problems and has distinguishing feature of high accuracy of the finite element methods. Because of this natural parallelism, they are of much current interest and have gigantic applied values[6]. Characteristic finite element method can use a larger time step to calculate and has very high accuracy. In present paper, we use a kind of transformation, the alternating-direction implicit characteristic finite element method is given for the initial and boundary value problem of the three-dimensional generalized nerve conduction equation. The primary advantage of this scheme is that: first, it involves only two time levels. Second, the initial conditions can be determined in a more natural way than that which is required in Douglas and Dupont's method. Third, the estimate of u_t is obtained at the same time. Because u_t is an important physical parameter, this scheme avoids arising two times error by using the commmon alternating-direction implicit characteristic finite element method to approximate u at first, then to approximate u_t. But this scheme yields the difficulty of error estimates of system form. Using Crank-Nicolson scheme for the transformation term, different technique of reducing error equation from the references, the calculation of vector product, H^{-1} norm estimates, the theory and technique of a priori estimate, we effectively overcome the difficulty of error estimates and optimal order error estimate in L^2 is obtained.

We make the following physical assumption:

$$u \in L^\infty(W^1_\infty)\cap H^1(H^{r+1}),\ u_t \in L^\infty(H^{r+1})\cap L^\infty(W^1_\infty)\cap H^2(L^2),\ u_{tt} \in L^2(H^{r+1}) \quad (1.2)$$

$f(s), g(l)$ and $b_i(x, s, l)(i = 1, 2, 3)$ are bounded, and Lipschitz continuous

with respect to s and 1, respectively. (1.3)

The letter M will be a generic constant and may be different each time when it is used. ε will be an arbitrarily small constant.

2. The Alternating-Direction Implicit Characteristic Finite Element Scheme

Let $v = u_t$, then (1.1a) can be rewritten as

$$v_t + b(x, u, v)\cdot\nabla v - \Delta v = H(u, v),\quad x \in \Omega,\ t \in J \quad (2.1\,a)$$

$$u_t = v,\quad x \in \Omega,\ t \in J \quad (2.1\,b)$$