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A MULTIPLICITY RESULT OF A SYSTEM OF VARIATIONAL INEQUALITIES

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Abstract A multiplicity result of a system of variational equalities, which is related to surfaces spanned over obstacles with prescribed mean curvature H, is obtained via a generalized Mountain Pass Lemma by proving a local compact result.
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1. Introduction and Statement of the Problem

The problem discussed in this paper is related to the surfaces with a prescribed mean curvature which are spanned over a given obstacle.

Let $B := \{z = (x,y) \in \mathcal{R}^2, x^2 + y^2 < 1\}, \Omega \subset B$ be a nonempty closed set, $\psi = (\psi^1, \psi^2, \psi^3) : B \longmapsto \mathcal{R}^3$ be a fixed function, $\gamma : \partial B \longmapsto \mathcal{R}^3$ be given. Let

$$\mathcal{M} := \{u \in H^{1,2}(B, \mathcal{R}^3), u \geq \psi \text{ a. e. in } \Omega, u|_{\partial B} = \gamma\}$$

where $u \geq \psi$ a.e. in Ω means that $u^i \geq \psi^i$ a.e. in Ω for $i = 1, 2, 3$. We consider the following

Problem 1 Find $u \in \mathcal{M}$, such that, for any $v \in \mathcal{M}$,

$$\int_B Du \cdot D(u-v)dz + 2H \int_B (u-v) \cdot u_x \wedge u_y dz \leq 0 \tag{1.1}$$

where $u_x = \dfrac{\partial u}{\partial x}, u_y = \dfrac{\partial u}{\partial y}$, the symbol $\wedge$ represents the exterior product.

The problem just described corresponds to a complementary boundary value problem:

Problem 2 Find $u \in C^0(\bar{B}, \mathcal{R}^3) \cap C^{1,1}(B, \mathcal{R}^3)$, such that

$$\begin{cases} \triangle u - 2H \cdot u_x \wedge u_y \leq 0 \text{ in } B \\ \triangle u^i - 2H \cdot (u_x \wedge u_y)^i = 0 \text{ in } B \backslash I_\Omega^i \\ u \in \mathcal{M} \end{cases} \tag{1.2}$$

where $I_\Omega^i := \{(x,y) \in \Omega, u^i(x,y) = \psi^i(x,y)\}$ and $i = 1,2,3$.

Actually one may easily prove (see [1]).

Theorem 1 *If $u \in C^0(\bar{B}, \mathcal{R}^3) \cap H^{2,2}(B, \mathcal{R}^3)$, then the complementary problem (1.2) (in the sense of distribution) and the problem of variational inequalities (1.3) are equivalent.*

In this paper we will confine ourselves on the multiplicity of the problem (1.1). The argument used is to prove that the functional associated has local compactness (cf.[2,3]) and then, on account of the topological assumption for the domain concerning the obstacle, at least two distinct critical points by virtue of the generalized mountain pass theorem (see [4]). The topological property which we suppose on the domain of the obstacle is related to the problem of hole and obstacle in [5]. Although we do not know if this kind of restriction is necessary for dealing with the multiplicity of H-surfaces spanned over an obstacle, a very deep result obtained by A. Bahri and J. M. Coron[6] which discussed the boundary value problem lack of compactness

$$\begin{cases} -\triangle u = u^{\frac{n+2}{n-2}} \text{ in } \Omega \subseteq \mathcal{R}^n \text{ open and bounded} \\ y \in H_0^1(\Omega) \end{cases} \tag{1.3}$$

demonstrated that the holes in the domain Ω may induce very rich topological properties about the sublevel sets of the energy affiliated (1.3).

Minimal surfaces and the surfaces with prescribed mean curvature have been very active and fruitful subjects in these years. These surfaces are intuitively known as soap films and soap bubbles. To find the H-surfaces u which satisfy the conformal conditions $|u_x|^2 - |u_y|^2 = u_x \cdot u_y = 0$ in B is so called the Plateau problem which is relevant to the Rellich's conjecture. An existence result was proved by Hildebrandt[7] for the case H =constant. The second solutions were obtained by Brezis and Coron[8], Strwue[2,9] (see also Steffen[10]). The regularity results on these kinds of problems may be found for examples [11–13]. More recently Bethuel and Rey[14], Jakobowsky[15] have proved the multiple solutions for the surfaces with nonconstant mean curvature. An existence result and some regularity for the minimal surfaces and H-surfaces spanned over obstacles have been established by Tomi in the paper [4].

2. The Small Solution

Let γ satisfy that $\gamma \geq \psi$ on $\Omega \cap \partial B$ (if nonempty),

$$\gamma \in H^{1/2}(\partial B; \mathcal{R}^3) \cap L^\infty(\partial B; \mathcal{R}^3)$$

$\psi^+ \in H^{2,p}(B; \mathcal{R}^3)$ (for some $p > 2$), here $\psi^+ := (\psi^{1+}, \psi^{2+}, \psi^{3+}), \psi^{i+} = \max\{0, \psi_i\}, i = 1,2,3.$