J.Partial Diff. Eqs. **13**(2000), 1–10
©International Academic Publishers Vol.13, No.1

SPECTRAL PROPERTIES OF SECOND ORDER DIFFERENTIAL OPERATORS ON TWO-STEP NILPOTENT LIE GROUPS*

Niu Pengcheng and Luo Xuebo
(Department of Applied Mathematics, Northwestern Polytechnical University
Xi'an 710072, China)
(Received July 12, 1997; revised Oct. 19, 1998)

Abstract In this paper, spectral properties of certain left invariant differential operators on two-step nilpotent Lie groups are completely described by using the theory of unitary irreducible representations and the Plancherel formulae on nilpotent Lie groups.

Key Words Spectrum; eigenvalue; left invariant differential operator; nilpotent Lie group.

1991 MR Subject Classification 35P20, 47F05.

Chinese Library Classification O175.3, O177.7.

1. Introduction

The purpose of this paper is to consider spectral properties of the following operators

$$P = -\sum_{r=1}^{p_1} X_r^2 + \sum_{l=1}^{p_2} C_l Y_l \tag{1.1}$$

where $X_r, Y_l, r = 1, \cdots, p_1, l = 1, \cdots, p_2$, are a basis for the Lie algebra $\mathcal{G}$ of a two-step nilpotent Lie group G, each C_l is a complex constant.

The operator P and its properties, e.g., local solvability, hypoellipticity, have been investigated by many authors. It is well known that (1.1) has not been contained in the class of operators introduced by Hörmander in [1] if the C_l is imaginary. Spectral properties of the Kohn-Laplacian on the Heisenberg group were studied by Luo and Niu in [2]. Also Furutani et al. discussed the spectrum of Laplacian on two-step nilpotent groups (See [3]).

In this paper we will determine the spectrum of P. The main tools here are unitary representations and Plancherel formulae on G.

In order to state our main results we need some notations. We denote the spectrum and the resolvant set of P by $\sigma(P)$ and $\rho(P)$ respectively. Let V_0 be the completion

* The project supported by National Natural Science Foundation of China.

of $C_0^\infty(G)$ with the norm (3.2) below and $D(P)$ the class of those $u \in V_0$ such that $Pu \in L^2(G)$. Introduce the set

$$\Gamma_\beta(C) = \Big\{ q \in C : q = \sum_{j=1}^{\frac{p_1-d}{2}} (2\beta_j+1)\rho_j + \sum_{i=1}^{d} \zeta_i^2 - \sqrt{-1}\sum_{l=1}^{p_2} C_l \eta_l, \quad \rho_j \in \mathbf{R}_+, \zeta_i \in \mathbf{R}, \eta_l \in \mathbf{R} \Big\} \tag{1.2}$$

where C denotes the complex plane, $C_l \in C$ $(l = 1, \cdots, p_2)$, $\frac{p_1-d}{2}$ is a positive integer, $\beta = (\beta_1, \cdots, \beta_{(p_1-d)/2})$, $\beta_j \in \mathbf{I}_+ = \{0, 1, 2, \cdots\}$, the definition of d will be given in Section 2, and

$$S(P) = \bigcup_{\beta \in I_+^{\frac{p_1-d}{2}}} \Gamma_\beta(C) \cup \mathbf{R}_+ \tag{1.3}$$

where $\mathbf{R}_+$ denotes the set of nonnegative real numbers.

Theorem 1.1 *The spectrum of P is $S(P)$.*

As a consequence, we have

Corollary 1.1 *If C_l $(l = 1, \cdots, p_2)$ is purely imaginary, then $\sigma(P)$ is either $\mathbf{R}$ or $[0, +\infty)$. In particular, if $C_l = 0$, $l = 1, \cdots, p_2$, then $\sigma(P) = [0, +\infty)$.*

Theorem 1.2 *If $d \neq 0$, then P has not any eigenvalue.*

Theorem 1.3 *If $d = 0$ and ρ_j, η_l, C_l, $j = 1, \cdots, \frac{p_1-d}{2}$; $l = 1, \cdots, p_2$, satisfy*

$$\sum_j (2\beta_j + 1)\rho_j - \sqrt{-1}\sum_l C_l \eta_l = 0 \tag{1.4}$$

for some $\beta \in \mathbf{I}_+^{\frac{p_1-d}{2}}$, then 0 is a unique eigenvalue of P.

The plan of this paper is as follows. In Section 2, we shall recall some basic results on the two-step nilpotent Lie group which will be used later. Section 3 is devoted to the proof of Theorem 1.1. In Section 4, we shall prove Theorems 1.2, 1.3. Finally in Section 5 some applications will be given.

2. Preliminaries

Let G be a connected, simply connected Lie group, whose Lie algebra $\mathcal{G}$ decomposes as a vector space $\mathcal{G} = \mathcal{G}_1 \oplus \mathcal{G}_2$ with $[\mathcal{G}_1, \mathcal{G}_1] \subset \mathcal{G}_2$, $[\mathcal{G}_1, \mathcal{G}_2] = \{0\}$. $\mathcal{G}$ carries a natural family of automorphic dilations given by

$$\delta_s(X) = sX \quad \text{if} \quad X \in \mathcal{G}_1, \quad \delta_s(Y) = s^2 Y \quad \text{if} \quad Y \in \mathcal{G}_2$$

These dilations extend in a natural way to $U(\mathcal{G})$, the universal envoloping algebra of $\mathcal{G}$, which may be identified with the set of all left invariant differential operators on G.