J.Partial Diff. Eqs. **13**(2000), 89–96
©International Academic Publishers Vol.13, No.1

HÖLDER ZYGMUND SPACE TECHNIQUES TO THE NAVIER-STOKES EQUATIONS IN THE WHOLE SPACES

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Dedicated to Professor Chen Wenyuan on his 70th birthday

(Received Apr.19, 1999)

Abstract With the use of Hölder Zygmund space techniques, local regular solutions to the Navier-Stokes equations in R^n are shown to exist when the initial data are in the space

$$\{a|(-\Delta)^{-\beta/2}a \in C^0(R^n)^n\} \quad (0 < \beta < 1)$$

Key Words Navier-Stokes equations; local solutions; Hölder Zygmund spaces.
1991 MR Subject Classification 35Q30, 76D05.
Chinese Library Classification O175.29.

1. Introduction

Consider the incompressible viscous fluid motion governed by the Navier-Stokes equations in R^n, $n \geq 2$:

$$\begin{cases} \partial u/\partial t - \Delta u + \nabla \cdot (u \otimes u) + \nabla \pi = 0 \\ \nabla \cdot u = 0 \\ u(0) = a \end{cases} \tag{1}$$

with unknown velocity $u = (u_1(x,t), \cdots, u_n(x,t))$ and unknown pressure $\pi = \pi(x,t)$. Here $\nabla =$ the gradient $(\partial_1, \cdots, \partial_n)$ and $\Delta =$ the Laplacian $\nabla \cdot \nabla$.

Mathematical theory of the Navier-Stokes equations stems from the poineering work of Leray [1] in 1934, where the existence of a global weak solution was established when the initial velocity $a \in L_2(R^n)^n$. The regularity of this weak solution, however, still remains foundamentally unknown. To understand the regularity problem, Fabes, Jones and Riviere [2] obtained the local existence of regular solutions with initial data in $L_p(R^n)^n$ with $n < p < \infty$ and the global existence of regular solutions with small initial data in $L_r(R^n)^n \cap L_p(R^n)^n$ with $1 \leq r < n < p < \infty$. This result has been extensively studied by many authors. For example, [3–7] and [8–9] are concerned with regular solutions when the initial velocity is in the Lebesgue space $L_p(R^n)^n$ with $p < \infty$ and the Lorentz space $L_{n,\infty}(R^n)^n$, respectively. It has become clear than $L_n(R^n)^n$ is a critical space in obtaining regularity solutions in the following sense: regular solution exists locally when the initial velocity $a \in L_p(R^n)^n$ with $n < p < \infty$, small regular solution exists globally when $a \in L_n(R^n)^n$, and no regular solution is found to exist when $a \in L_p(R^n)^n$ with $p < n$ no matter how small the $\|a\|_{L^p}$ is. One can also refer to [10–14] for stability study on fluid motions and [15–18] for bifurcation analysis of Navier-Stokes flows.

The purpose of this paper is to present a new approach showing the local existence of regular solutions when the initial data are in a new function space containing $L_p(R^n)^n$ with $n < p < \infty$.

To state our result, we denote by F the Fourier transform in R^n and set the Riesz potential $(-\Delta)^{\lambda/2} = F^{-1}|\xi|^\lambda F$. Moreover, we introduce the Hölder Zygmund space $\mathcal{C}^\alpha(R^n)$:

$$[u]_{\mathcal{C}^\alpha} \equiv \sup_{y\neq 0} \frac{\|u(\cdot+y)-u(\cdot)\|_{L_\infty}}{|y|^\alpha} \quad \text{for } 0<\alpha<1$$

$$[u]_{\mathcal{C}^0} \equiv [(-\Delta)^{-1/4}a]_{\mathcal{C}^{1/2}},\ [u]_{\mathcal{C}^\alpha} \equiv [(-\Delta)^{\alpha/2-1/4}a]_{\mathcal{C}^{1/2}} \quad \text{for } \alpha \geq 1$$

$$\mathcal{C}^\alpha(R^n) \equiv \begin{cases} \{u \in L_\infty(R^n) \mid \|u\|_{\mathcal{C}^\alpha} \equiv \|u\|_{L_\infty} + [u]_{\mathcal{C}^\alpha} < \infty\} & \text{for } \alpha > 0 \\ \{u \in S'(R^n) \mid \|u\|_{\mathcal{C}^0} \equiv [u]_{\mathcal{C}^0} < \infty\} & \text{for } \alpha = 0 \end{cases}$$

where $S'(R^n)$ denotes the dual space of $S(R^n)$, the Schwartz space of repidly decreasing smooth scalar functions.

The main result of this paper reads as follows:

Theorem 1.1 *Let $n \geq 2$, $0 < \beta < 1$, $(-\Delta)^{-\beta/2}a \in \mathcal{C}^0(R^n)^n$ and $\nabla \cdot a = 0$ in the sense of distribution. Then there exists a constant $T > 0$ such that Eq. (1) admits a regular solution u satisfying*

$$(-\Delta)^{-\beta/2}u \in C_{w-*}([0,T];\mathcal{C}^0(R^n)^n)$$

and

$$\|(-\Delta)^{-\beta/2}u(t)\|_{\mathcal{C}^0} + t^{\beta/2}\|u(t)\|_{L_\infty} + t\|u(t)\|_{\mathcal{C}^{2-\beta}} \in L_\infty(0,T)$$

where C_{w-*} denotes the continuity in the weak$-*$ topology.

Theorem 1.1 is to be proved in Section 2 based on elementary properties of the Hölder Zygmund spaces described in Section 2.

Let us mention that Giga, Inui and Matsui [19] recently obtained the local existence of regular solutions with initial data in $L_\infty(R^n)^n$ together with its subspaces. However, our study ion is rather different from those of [19] due to the fact that Theorem 1.1 shows the sharp regularity estimate in Hölder Zygmund spaces and the initial data

$$a \in \{a \in S'(R^n)^n \mid (-\Delta)^{-\beta/2}a \in \mathcal{C}^0(R^n)^n\}$$

which contains $L_p(R^n)^n$ with $p = n/\beta$, by the homogeneity and the Sobolev imbedding theorem (See [20]).