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SCATTERING FOR SEMILINEAR WAVE EQUATION WITH SMALL DATA IN HIGH SPACE DIMENSIONS

Yang Han
(Dept. of Applied Mathematics, Southwest Jiaotong University, Chengdu 610031, China)
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Abstract In this paper we study the scattering theory for the semilinear wave equation $u_{tt}-\Delta u = F(u(t,x), Du(t,x))$ in $\mathbf{R}^n$ $(n \geq 4)$ with smooth and small data. We show that the scattering operator exists for the nonlinear term $F = F(\lambda) = O(|\lambda|^{1+\alpha})$, where α is an integer and satisfies $\alpha \geq 2, n = 4$; $\alpha \geq 1$, $n \geq 5$.

Key Words Semilinear wave equation; scattering; asymptotic behavior.
Classification 35B40, 35P25.

1. Introduction

We consider the scattering problem for the semilinear wave equation

$$u_{tt}(t,x) - \Delta u(t,x) = F(u(t,x), Du(t,x)) \tag{1.1}$$

Here $Du(t,x) = (u_t, \nabla u)(t,x)$. Let $\lambda = (\lambda_0, (\lambda_i), i = 1, \cdots, n+1)$, suppose that in a neighbourhood of $\lambda = 0$, say, for the nonlinear term in (1.1) is a sufficiently smooth function satisfies

$$F(\lambda) = O(|\lambda|^{1+\alpha}) \tag{1.2}$$

where α is an integer ≥ 1. From Li and Yu [1], we know that if there is a relation between α and n as follows

$$\alpha \geq 2, \quad n = 4; \quad \alpha \geq 1, \quad n \geq 5 \tag{1.3}$$

then when initial data is sufficiently small, the Cauchy problem of (1.1) admits a unique global solution to $t \geq 0$. In this paper we establish the scattering theory for (1.1) under the same assumptions on α and n.

In the scattering theory the asymptotic behavior of the solution to (1.1) is compared with that of the solution to the free wave equation

$$u_{tt}(t,x) - \Delta u(t,x) = 0 \tag{1.4}$$

in the energy norm. More precisely, let $E(\mathbf{R}^n)$ be the energy space

$$E(\mathbf{R}^n) = \{(f,g); f \in L^2(\mathbf{R}^n), \nabla f, g \in L^2(\mathbf{R}^n)\}$$

with the norm $\|(f,g)\|_e = \|\nabla f\|_{L^2} + \|g\|_{L^2}$. We show that there exists a suitable normed space Σ dense in $E(\mathbf{R}^n)$ and a neighbourhood $\mathcal{N}$ of the origin in Σ such that for any $(f^-, g^-) \in \mathcal{N}$, there exists a unique solution $u(t,x)$ to (1.1) in $\mathbf{R} \times \mathbf{R}^n$, which behaves asymptotically like the solution $u_0^-(t,x)$ of the Cauchy problem (1.4) with data (f^-, g^-) at $t = 0$ in the sense that

$$\|u(t,\cdot) - u_0^-(t,\cdot)\|_e \to 0 \quad as \quad t \to -\infty$$

and moreover, we also find a unique solution $u_0^+(t,x)$ to (1.4) with corresponding data (f^+, g^+) at $t = 0$ such that

$$\|u(t,\cdot) - u_0^+(t,\cdot)\|_e \to 0 \quad as \quad t \to +\infty$$

Therefore, the scattering operator $S : (f^-, g^-) \to (f^+, g^+)$ can be defined in the set $\mathcal{N}$, provided that the conditions concerning α and n as in (1.3) are satisfied.

For the case that the nonlinear term F doesn't depend on $Du : F(u) = C|u|^p$ with smooth and small data most of the results were known. One can see Reed [2], Strauss [3], Pecher [4, 5], Mochizuki and Motai [6], Morawetz and Strauss [7] and Tsutaya [8]. For the case of large data, we refer the reader to the results by Ginibre and Velo [9]. Recently Kubo and Kubota [10] have considered the asymptotic behavior of the radial solution to (1.1). But for the generalized case as in (1.2), there are few results as the author knows.

In order to establish the scattering theory for (1.1), as usually, we start with considering the following Yang-Feldman equation [11]

$$u(t) = u_0(t) + \int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))d\tau \tag{1.5}$$

in a suitable invariable Sobolev space, here $\omega = (-\Delta)^{\frac{1}{2}}$. However, (1.5) is not useful directly to our problem, because we employ the generalized Sobolev space with weights related to the generators of Lorentz group. More precisely, since we apply a set of partial differential operators which have the weights x and (or) t, it is not clear whether the operators commute with the integral sign in (1.5). Thus one of our main tasks is to investigate the commutation relations between the partial operators and $\int_{-\infty}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))\, d\tau$. For that purpose we investigate the approximated solution

$$u_\sigma(t) = u_0(t) + \int_{\sigma}^{t} \frac{\sin\omega(t-\tau)}{\omega} F(u(\tau), Du(\tau))d\tau \tag{1.6}$$