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A HOPF BIFURCATION IN A GENERALIZED MCKEAN TYPE OF FREE BOUNDARY PROBLEM SATISFYING THE DIRICHLET BOUNDARY CONDITION *

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Abstract In this paper, we consider the free boundary problem satisfying the Dirichlet boundary condition. This problem is derived from the reaction diffusion equations with the generalized McKean reaction dynamics. We shall show a Hopf bifurcation occurs at some critical point τ when the stationary solution $(v^*(x), s^*)$ satisfies $1/3 < s^* < 1$.

Key Words Evolution equation; free boundary problem; Hopf bifurcation; internal layer solution; parabolic equation.

Classification 35R35, 35B32, 35B25, 35K22, 35K57, 58F14, 58F22.

1. Introduction

The well posedness and the Hopf bifurcation in a parabolic free boundary problem with the McKean reaction term are proved in [1]. In this paper, we consider the free boundary problem for the generalized McKean kinetics satisfying the Dirichlet boundary condition. We are dealing with the following problem.

$$\begin{cases} v_t = Dv_{xx} - (1+b)v + H(x - s(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+ \\ v(0,t) = 0 = v(1,t) & \text{for } t > 0 \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1 \\ \tau \dfrac{ds}{dt} = C(v(s(t),t)) & \text{for } t > 0 \\ s(0) = s_0 & \end{cases} \tag{1}$$

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where $v(x,t)$ and $v_x(x,t)$ are assumed continuous in $\Omega, \Omega = (0,1) \times (0,\infty)$. Here $H(y)$ is the Heaviside function. The velocity of the free boundary $s(t)$ is defined by

$$C : I \to \mathbf{R}, \quad I = \left(-\frac{a}{2}, \frac{1-a}{2}\right)$$

and the explicit form of C is given by

$$C(v) = \frac{2v - \frac{1-2a}{2}}{\sqrt{(\frac{1-a}{2} - v)(v + \frac{a}{2})}}, \quad 0 < a < 1$$

This problem has its origins in some work by P. Fife ([2], [3]) on asymptotic analysis of the dynamics of internal layers in reaction diffusion equations. The free boundary problem is an outgrowth of work done by M. Mimura, Y. Nishiura and their coworkers ([4–7]). These authors take as a starting point a system of two reaction diffusion equations

$$\varepsilon\tau u_t = \varepsilon^2 u_{xx} + f(u,v), \quad v_t = Dv_{xx} + g(u,v) \tag{2}$$

depending on two small parameters ε, τ. Here u and v measure the levels of two diffusing quantities. The functions u and v are assumed to satisfy Dirichlet boundary conditions at $x = 0, 1$. The functions f and g are assumed to be of bistable type, i.e., the equation $f = 0$ determines u as a triple valued function of v and the curves defined by $f = 0$, $g = 0$ have three points of intersection, which determine all of the interactions between u and v. The term bistable refers to the fact that these points of intersection correspond to equilibria of the system (2), two of which are stable, the third unstable.

When ε and τ are chosen to be very small, the system (2) models a situation in which the quantity measured by u reacts much faster than that measured by v (τ small), while at the same time u diffuses slower than v (ε small). The principal interest in systems like (2) comes from the fact that there exist families of stationary solutions parametrized by ε, which approach discontinuous functions of x as $\varepsilon \to 0$. When ε is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place within an x-interval of length $O(\varepsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called a layer – a boundary layer when it is adjacent to an endpoint of the interval or an internal layer when it is in the interior of the interval.

In 1981, Mimura, Tabata and Hosono ([4]) proved the existence of nontrival internal layer solutions to the stationary (time-independent) problem associated with (2). The question of the stability of these stationary layer solutions when ε is small was later dealt with in a pair of papers; one by Nishiura and Fujii ([5]) appearing in 1987 for the