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CAUCHY'S PROBLEM FOR DEGENERATE QUASILINEAR HYPERBOLIC EQUATIONS WITH MEASURES AS INITIAL VALUES

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Abstract The aim of this paper is to discuss the Cauchy problem for degenerate quasilinear hyperbolic equations of the form

$$\frac{\partial u}{\partial t}+\frac{\partial u^m}{\partial x}=-u^p, \quad m>1,\ p>0$$

with measures as initial conditions. The existence and uniqueness of solutions are obtained. In particular, we prove the following results:

(1) $0<p<1$ is a necessary and sufficient condition for the above equations to have extinction property;

(2) $0<p<m$ is a necessary and sufficient condition for the above equations to have localization property of the propagation of perturbations.

Key Words Degenerate quasilinear hyperbolic equations; existence and uniqueness; extinction and positivity; localization.

Classification 35L80, 35L60, 35L15, 35B40, 35F25.

1. Introduction

In this paper we consider the degenerate quasilinear hyperbolic equations of the form

$$\frac{\partial u}{\partial t}+\frac{\partial u^m}{\partial x}=-u^p \tag{1.1}$$

in $Q\equiv \mathbf{R}\times(0,+\infty)$ with the following initial conditions

$$u(x,0)=\mu(x) \tag{1.2}$$

for all $x\in\mathbf{R}$, where $m>1$, $p>0$, and μ is a nonnegative non-zero finite Borel measure in $\mathbf{R}$, and $\mathbf{R}=(-\infty,+\infty)$.

The equation (1.1) appears in the theory of traffic flow with the velocity mu^{m-1}, where u is the density of the traffic flow in general, see M.J. Lighthill and G.B. Whitham [1], P.I. Richards [2], G.B. Whitham [3]. It is degenerate at $u=0$, and therefore has no classical solution in general. We consider its generalized solutions.

Definition 1.1 *A nonnegative function $u : Q \mapsto \mathbf{R}$ is said to be a solution of* (1.1), *if $u \in L^\infty(0,+\infty; L^1(\mathbf{R})) \cap C(0,+\infty; L^1(\mathbf{R}))$ satisfies the following conditions:*

(i) *For any $\tau \in (0,+\infty)$, we have $u \in L^\infty(\mathbf{R} \times (\tau,+\infty))$;*

(ii) *For any $\phi \in C_0^\infty(Q)$ with $\phi \geq 0$, we have*

$$\iint_Q \operatorname{sign}(u-k)\Big[(u-k)\frac{\partial \phi}{\partial t} + (u^m - k^m)\frac{\partial \phi}{\partial x} - u^p \phi\Big] dxdt \geq 0, \quad \forall k \in \mathbf{R}$$

where $\operatorname{sign}(u-k) = 1$ *if* $u > k$; $\operatorname{sign}(u-k) = -1$ *if* $u < k$; $\operatorname{sign}(u-k) = 0$ *if* $u = k$.

Definition 1.2 *A nonnegative function $u : Q \mapsto \mathbf{R}$ is said to be a solution of the Cauchy problem* (1.1)–(1.2), *if u is a solution of* (1.1) *and satisfies the initial conditions* (1.2) *in the sense of the distribution:*

$$\operatorname{ess}\lim_{t\to 0^+} \int_{\mathbf{R}} \psi(x)u(x,t)dx = \int_{\mathbf{R}} \psi d\mu, \quad \forall \psi \in C_0^\infty(\mathbf{R}).$$

For $\mu \in L^\infty(\mathbf{R})$, the Cauchy problem (1.1)–(1.2) has been studied by a number of authors, cf. Guiqiang Chen [4], E. Conway and J. Smoller [5], R. Courant [6], R. Courant and K. Friedrichs [7], R. Courant and P.D. Lax [8], M.G. Crandall and A. Majda [9], C. Dafermos [10], R. DiPerna [11, 12], J. Glimm [13], S. Kruzkov [14], A.I. Vol'pert and S.I. Hudjaev [15], A.I. Vol'pert [16], Wu Zhuoqun and Yin Jingxue [17].

Our main results are the following theorems.

Theorem 1.1 (Existence) *Let $0 < p < m+1$. Then the Cauchy problem* (1.1)–(1.2) *has at least one solution.*

Such existence of solutions for the equation

$$\frac{\partial u}{\partial t} - \frac{\partial \phi(u)}{\partial x} = 0$$

has been obtained by T.P. Liu and M. Pierre [18] under suitable assumptions on ϕ. Recently, F.R. Guarguaglini [19] has constructed a solution for the equation of the form

$$\frac{\partial u}{\partial t} - \frac{\partial u^m}{\partial x} = -u^p, \quad m > 1,\ p > 1 \tag{1.3}$$

with $\delta(x)$ as initial value, where $\delta(x)$ is the Dirac measure at the origin in $\mathbf{R}$. But, the proofs in [18] and [19] are based on the regularizing effects in [20] and [21], and can not be applied to our case $p > 0$. Our method is different from [18] and [19], it is based on some local BV_t-estimates and some local BV_x-estimates (See Proposition 4.2–4.4).

Remark 1.1 In the case $p \geq m+1$ and $\mu(x) = \delta(x)$, F.R. Guarguaglini proved that the Cauchy problem (1.1)–(1.2) has no solution (See [19]). This implies that the conclusion of Theorem 1.1 is optimal.

Theorem 1.2 (Uniqueness) *Let $0 < p < m+1$. Then the Cauchy problem* (1.1)–(1.2) *has at most one solution.*

Such uniqueness has been obtained by F.R. Guarguaglini [19] for the equation (1.3) with $\delta(x)$ as initial value. But, the proof in [19] is based on the symmetric property