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INTERFACE PROBLEMS FOR ELLIPTIC SYSTEMS

Wu Jinbiao
(School of Mathematics Sciences, Peking University, Beijing 100871, China)
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Abstract Interface problems for elliptic systems of second order partial differential equations are studied. The main result is that the solution in the neighborhood of the singular point can be divided into two parts one of which is a solution to the homogeneous system with constant coefficients, and the other one possesses higher regularity.
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1. Introduction

Consider the following elliptic system of second order partial differential equations with discontinuous coefficients

$$-D_\alpha(A_{ij}^{\alpha\beta}(x)D_\beta U^j) = F_i, \quad i = 1, \cdots, N, x \in \Omega \tag{1}$$

where $\alpha, \beta = 1, 2$, $j = 1, \cdots, N$, $N \geq 2$, $\Omega \subset R^2$ is a polygonal domain, and the summation convention is assumed. We assume that Ω is decomposed into a finite number of polygonal subdomains $\Omega^{(k)}$, such that $\cup\overline{\Omega^{(k)}} = \overline{\Omega}$, $A_{ij}^{\alpha\beta}(x) \in W^{1,\infty}(\Omega^{(k)})$. Let $H^s(\Omega, R^N) = \{V = (V^1, \cdots, V^N) | V^i \in H^s(\Omega), 1 \leq i \leq N\}$, with the norm $\|U\|_{s,\Omega} = \Big(\sum_{i=1}^N \|U^i\|^2_{H^s(\Omega)}\Big)^{1/2}$, for $U \in H^s(\Omega, R^N)$ $(s = 0, 1, 2)$, and $|\cdot|_{s,\Omega}$ denote the corresponding semi-norms.

Let $F = (F_1, \cdots, F_N)$, $U = (U^1, \cdots, U^N)$. Now we assume $F \in L^2(\Omega, R^N)$, and try to discuss the regularity of the weak solution U to (1). We will prove that $U \in H^2(\Omega' \cap \Omega^{(k)}, R^N)$ where Ω' contains no singular point, from the proof in Section 2, then we mainly study the local behavior of U near singular points. For elliptic system the elasticity problem is a significant example. In this paper, we study the general case of elliptic system with the background of the elasticity problem.

Consider the plain elasticity problem [1]:

$$\begin{cases} -D_1((\lambda + 2\mu)D_1U^1 + \lambda D_2U^2) - D_2(\mu D_2U^1 + \mu D_1U^2) = F_1 \\ -D_1(\mu D_2U^1 + \mu D_1U^2) - D_2(\lambda D_1U^1 + (\lambda + 2\mu)D_2U^2) = F_2 \end{cases}$$

where λ, μ are piecewise constant functions, and they are lamè constants. This system doesn't satisfy the strong Legendre condition,

$$A_{ij}^{\alpha\beta}\xi_\alpha^i\xi_\beta^j \geq \chi|\xi|^2, \quad \forall \xi \in R^{2\times N} \tag{2}$$

where $\chi > 0$ is a constant. But applying the Korn inequality [2]:

$$\|U\|_{1,\Omega}^2 \leq C(\Omega)\Big(\int_\Omega (D_1U^1)^2 + (D_2U^2)^2 + \Big(\frac{D_1U^2 + D_2U^1}{2}\Big)^2 dx + \|U\|_{0,\Omega}^2\Big)$$

for $U \in H^1(\Omega, R^2)$, we deduce that

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta U^j D_\alpha U^i dx \geq C\int_\Omega |DU|^2 dx, \quad \forall U \in H_*^1(\Omega, R^2) \tag{3}$$

where $H_*^1(\Omega, R^2) = \{U | U \in H^1(\Omega, R^2),\ U|_{\Gamma_0} = 0\}$, $\Gamma_0 \subset \partial\Omega$ with a strictly positive measure. If we assume $A_{ij}^{\alpha\beta}$ satisfy the elliptic condition (3), we can also discuss kinds of boundary condition (We only try to know the regularity of the solution near the singular point, so without losing generality we assume that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, where on Γ_0 we impose the Dirichlet condition). And the argument is analogous.

So for simplicity we impose the Dirichlet boundary condition

$$U|_{\partial\Omega} = 0 \tag{4}$$

on (1), and assume $A_{ij}^{\alpha\beta}$ satisfy

$$\int_\Omega A_{ij}^{\alpha\beta} D_\beta U^j D_\alpha U^i dx \geq \chi_0 \int_\Omega |DU|^2 dx, \ \forall U \in H_0^1(\Omega, R^N) \tag{5}$$

where χ_0 is a positive constant.

Our main result is the following theorem.

Theorem 1 *We assume that $F \in L^2(\Omega, R^N)$, $A_{ij}^{\alpha\beta}$ satisfy (5), $\tilde{x}$ is a singular point. Let $\tilde{\Omega}$ be a neighborhood of $\tilde{x}$ which contains the only singular point $\tilde{x}$, and U be the weak solution to* (1) (4). *We have*

$$U = V + W$$

in $\tilde{\Omega}$, where $V \in H^1(\tilde{\Omega}, R^N)$ and satisfies the system

$$-D_\alpha(A_{ij}^{\alpha\beta}(\tilde{x}) D_\beta U^j) = 0 \tag{6}$$

and the boundary condition (4) *if $\tilde{x} \in \partial\Omega$, where $A_{ij}^{\alpha\beta}(\tilde{x})$ are piecewise constant coefficients frozen in $\tilde{x}$ and we obtain the following estimate*

$$\|V\|_{1,\tilde{\Omega}} + \|W\|_{1,\tilde{\Omega}} + \Big\|\frac{D^2W}{(|\log r| + 1)^M}\Big\|_{0,\tilde{\Omega}\cap\Omega^{(k)}} \leq C(\|U\|_{1,\Omega} + \|F\|_{0,\Omega}) \tag{7}$$