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COLORED BLACK HOLES

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Dedicated to Professor Ding Xiaxi on the occasion of his 70th birthday

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Abstract We describe some recent results on solutions of the static, stationary, spherically symmetric solutions of the $SU(z)$ Einstein-Yang/Mills equations. The main result is that any solution which is defined in the far field and has finite (ADM) mass, is defined for all $r > 0$.

Key Words Einstein-Yang-Mills equations; colored black holes; event horizon.
Classification 83C15, 83C57, 83C20, 58E15.

1. In this paper we describe some recent results on spherically symmetric black hole solutions of the $SU(2)$ Einstein-Yang/Mills (EYM) equations; such solutions are called "colored black holes." Our main result is that given any solution to the EYM equations which is defined in the far field ($r \gg 1$) and has finite (ADM) mass, is defined for all $r > 0$; see [1]. Since we know (see [2, 3, 4]) that given any event horizon, there are an infinite number of black-hole solutions having event horizon ρ, our result implies that all of these solutions can be continued back to zero. In particular this gives information as to the behavior of the gravitational field and the Yang-Mills field, inside a black hole, a subject of recent interest; [5, 6].

Our main result is surprising since, generally speaking, for nonlinear equations, existence theorems are only local, with perhaps global existence only for special parameter values. However for the EYM equations we prove global existence for any solution which is defined in a neighborhood of infinity. This result is not true in the "other direction"; namely if a solution is defined near $r = 0$, with particle-like boundary conditions (see [7]), a singularity can develop at some $\overline{r} > 0$, and the solution cannot be extended beyond $r = \overline{r}$ (see [8, Thm. 4.1]).

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2. The coupled EYM equations in 3 + 1 space time with gauge group G, can be written in the form [1-20]

$$R_{ij} - \frac{1}{2}Rg_{ij} = \sigma T_{ij}, \quad d^*F_{ij} = 0, \quad i,j = 0,1,2,3 \tag{1}$$

Here $R_{ij} - \frac{1}{2}Rg_{ij}$ is the Einstein tensor computed with respect to the (unknown) metric g_{ij} and T_{ij} is the stress-energy tensor associated to the g-valued Yang/Mills curvature 2-form F_{ij} where g is the Lie algebra of G. If $G = SU(2)$, and we seek static, spherically symmetric solutions (solutions depending only on r), then we may write the metric as

$$ds^2 = -AC^2dt^2 + A^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

and the YM curvature 2-form as [7],

$$F = w'\tau_1 dr \wedge d\theta + w'\tau_2 dr \wedge (\sin\theta d\phi) - (1-w^2)\tau_3 d\theta \wedge (\sin\theta d\phi) \tag{3}$$

Here (A, C), and w denote the unknown metric and connection coefficients respectively, and τ_1, τ_2 and τ_3 form a basis (the Paul matrices) for the Lie algebra $su(2)$. As has been discussed elsewhere, (cf. [1-20]), it follows from (1)–(3), that the spherically symmetric $SU(2)$ EYM equations are

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1-w^2)^2}{r^2} \tag{4}$$

$$r^2A'' + \left[r(1-A) - \frac{(1-w^2)^2}{r^2}\right]w' + w(1-w^2) = 0 \tag{5}$$

and

$$\frac{C'}{C} = \frac{2w'^2}{r} \tag{6}$$

Notice that (4) and (5) don't involve C, so the major part of our effort is to solve (4) and (5).

A (colored) black-hole solution of (5)–(6) having event horizon $\rho > 0$ is a solution defined for all $r > \rho$ and satisfying

$$A(\rho) = 0, \quad A(r) > 0 \quad \text{if } r > \rho \tag{7}$$

$$\lim_{r\to\infty} \mu(r) \equiv \lim_{r\to\infty} r(1 - A(r)) = \overline{\mu} < \infty \tag{8}$$

and

$$\lim_{r\to\infty} (w^2(r), w'(r)) = (1, 0) \tag{9}$$

These conditions imply that the metric is asymptotically flat and has finite (ADM) mass $\overline{\mu}$, and that the YM field is well-behaved.