J.Partial Diff. Eqs. **11**(1998), 125–136
©International Academic Publishers Vol.11, No.2

THE STABILITY OF NAVIER-STOKES EQUATIONS AND THE ESTIMATION OF ITS ATTRACTOR DIMENSION*

Li Kaitai and Hu Changbing
(Research Center of Applied Mathematics, Xi'an Jiaotong University, 710049)
(Received Nov. 8, 1994; revised Mar. 27, 1997)

Abstract In this paper, for a class of exterior force term $2s^2W'_{(s,s)}$, we analyse the existence of unstable modes of linearized Navier-Stokes Equations (NSE), and associate them with integer points in plane. Furthermore we give the lower boundary dimension estimation of the attractor of NSE. Liu discussed the condition where the exterior force term is $W'_{(0,s)}$ in [1, 2], but his method can't be extended to the condition where the exterior force term is $W'_{(s_1,s_2)}$ ($s_1 \neq 0$, $s_2 \neq 0$). So this paper may look as the extention of [1, 2]. The method which we give in this paper has direct application for further study of other properties of NSE (such as Hopf bifurcation). See [3].
Key Words Navier-Stokes equations; attractor; unstability
Classification 35Q30, 76D05.

1. Introduction

The attractor of NSE and its dimension estimation is a practical and profound dynamic problem, and it has direct relation with the turbulent structure. The study of NSE first connected the two kinds of turbulent methods together [4]: Kolmogorov method and dynamic theory method. And the approximate study for attractors led two new concepts: Inertial Manifolds (IMs) and Approximate Inertial Manifolds (AIMs). They are all the most active problems in dynamic theory.

In the following, we consider NSE with periodic boundary conditions confined on $\Omega = [0, 2\pi] \times [0, 2\pi]$:

$$\begin{cases} \dfrac{du}{dt} + Au + B(u, u) = f \\ u(0) = u_0 \end{cases} \tag{1.1}$$

* The research supported by State Key Major Preject of Basic Research and NSF of China.

where $u \in H$,H is a Hilbert space and the functions in it satisfy:

$$\begin{cases} u = \sum_{j=(j_1,j_2)\in Z^2} u_j e^{ij\cdot x}, \quad u_j \in C^2, \quad u_{-j} = \overline{u}_j \\ u_0 = 0, \quad j \cdot u_j = 0, \quad |u|^2 = (2\pi)^2 \sum_{j\in Z^2} |u_j|^2 < +\infty \end{cases} \tag{1.2}$$

here $j \cdot x = j_1 x_1 + j_2 x_2$, $u_j = (u_j^1, u_j^2)$, $j \cdot u_j = j_1 u_j^1 + j_2 u_j^2$. $|u|^2$ is the norm of H.

Denote $P : L^2(\Omega) \to H$ as the Helmholtz projector. Then:

$$A(u) = -P\Delta u, \quad B(u,v) = P(u \cdot \nabla)v$$

For any $k = (k_1, k_2) \neq (0,0)$, set

$$W_k = \frac{1}{\sqrt{2}\pi|k|} k' \cos k \cdot x, \quad W_k' = \frac{1}{\sqrt{2}\pi|k|} k' \sin k \cdot x$$

It is well known that W_k, W_k' are eigenvectors of A corresponding to the eigenvalue $|k|^2$ under the periodic boundary conditions, where $k' = (k_2, -k_1)$, $|k| = \sqrt{k_1^2 + k_2^2}$. $k \cdot x = k_1 x_1 + k_2 x_2$.

For the given exterior force term $f_0 = 2s^2 \lambda W'_{(s,s)}$, there exists the stable solution $u_0 = \lambda W'_{(s,s)}$. λ is the Reynolds number, and then the dimensionless Grashof number is defined as:

$$G = \frac{|f|}{\nu^2 \lambda_1} = 2s^2\lambda \quad (\nu = 1, \lambda_1 = 1)$$

Now the NSE linearized at point u_0 reads:

$$\begin{cases} \frac{dw}{dt} + A(u_0)w = 0 \\ A(u_0)w = Aw + B(w, u_0) + B(u_0, w) \end{cases} \tag{1.3}$$

We will discuss the following eigenvalue problem:

$$A(u_0)V = -\sigma V \tag{1.4}$$

For $\mathrm{Re}\,\sigma > 0 (= 0, < 0)$, the related V is called steady (neutral, unsteady) modes. In this paper, all possible unsteady modes correspond to the integer points set in plane. From the conclusion in [5], we obtain the lower boundary of the dimension of the attactors of NSE.

2. Conceptions and Preparing Theorems

Differing from [1], for the problem (1.4), $\mathcal{K}$ is defined as the integer points set in plane (k_1, k_2).