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GLOBAL EXISTENCE AND BLOW-UP FOR A PARABOLIC SYSTEM WITH NONLINEAR BOUNDARY CONDITIONS

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Abstract This paper deals with the global existence and blow-up of positive solutions to the systems:

$$\begin{aligned} &u_t = \nabla(u^m \nabla u) + u^l + v^a \\ &v_t = \nabla(v^n \nabla v) + u^b + v^k \quad \text{in} \quad B_R \times (0,T) \\ &\frac{\partial u}{\partial \eta} = u^\alpha v^p, \quad \frac{\partial v}{\partial \eta} = u^q v^\beta \quad \text{on} \quad S_R \times (0,T) \\ &u(x,0) = u_0(x), \quad v(x,0) = v_0(x) \quad \text{in } B_R \end{aligned}$$

We prove that there exists a global classical positive solution if and only if $l \le 1$, $k \le 1$, $m + \alpha \le 1$, $n + \beta \le 1$, $pq \le (1 - m - \alpha)(1 - n - \beta)$,$ab \le 1$, $qa \le (1 - n - \beta)$ and $pb \le (1 - m - \alpha)$.

Key Words Parabolic system; global existence; blow-up.
Classification 35K50, 35K60.

1. Introduction

In this paper, we study the positive solution to the systems:

$$\begin{aligned} &u_t = \nabla(u^m \nabla u) + u^l + v^a \\ &v_t = \nabla(v^n \nabla v) + u^b + v^k \quad \text{in} \quad B_R \times (0,T) \\ &\frac{\partial u}{\partial \eta} = u^\alpha v^p, \quad \frac{\partial v}{\partial \eta} = u^q v^\beta \quad \text{on} \quad S_R \times (0,T) \\ &u(x,0) = u_0(x), v(x,0) = v_0(x) \quad \text{in} \quad B_R \end{aligned} \tag{1.1}$$

where $B_R = \{|x| < R\}$ in $\mathbb{R}^N$, $S_R = \{|x| = R\}$, $\partial u/\partial \eta$ is the derivative of u in the direction of the outward normal, and $m, n, l, k, a, b, p, q, \alpha, \beta$ are nonnegative real numbers, and u_0, v_0 are smooth positive functions satisfying compatibility conditions:

$$\frac{\partial u_0}{\partial \eta} = u_0^\alpha v_0^p, \quad \frac{\partial v_0}{\partial \eta} = u_0^q v_0^\beta \quad \text{on} \quad S_R \tag{1.2}$$

In the past few years, many works have been done to the study of the precise behavior of solutions to a single equation

$$u_t = \Delta u \quad \text{in} \quad \Omega \times (0, T)$$

with a nonlinear boundary condition

$$\frac{\partial u}{\partial \eta} = f(u) \quad \text{on} \quad \partial\Omega \times (0, T)$$

here Ω is a bounded domain in $\mathbb{R}^N$. It is known that for each f the existence of global solution only depends on the behavior of f at infinity. Walter [1] posed conditions on f that ensured finite time blow-up for large initial data and global existence for small ones. Gomez et al. [2] proved that if $\Omega = B_R$ and f is nondecreasing with $1/f$ integrable at infinity, nonnegative solutions with any initial data can exist locally. If $f(u)$ is replaced by u^p, other results of this kind can be seen in [3] or [4]. For the nonlinear parabolic equations, the behavior of solutions were studied by a number of authors (cf. [5–8], etc.).

Equation (1.1) constitutes a simple example of a semilinear reaction-diffusion system exhibiting a nontrivial coupling on the unknown $u(x, t)$, $v(x, t)$. They can be used as a model to describe heat propagation in a two-component combustible mixture. In this case u and v represent the temperatures of the interacting components, the energy release given by some power of u and v in the right-hand side of (1.1) is assumed.

Recently some important results have been obtained on the large time behavior of solution to parabolic systems with nonlinear boundary condition. Deng [9] gave the necessary and sufficient condition for global existence in the special case of radial solutions. For a more general case Wang [10] also gave the necessary and sufficient conditions.

This work has been motivated by the results of [9] considering a system of heat equations $u_t - \Delta u = 0$, $v_t - \Delta v = 0$ with nonlinear boundary conditions $\partial u/\partial\eta = v^p$, $\partial v/\partial\eta = u^q$ and by the result of [11] considering the system of semilinear parabolic equations $u_t = \alpha_1 u_{xx} + v^p$, $v_t = \alpha_2 v_{xx} + u^q$ with Neumann conditions $\partial u/\partial\eta = \partial v/\partial\eta = 0$. We know the effect of v^p and u^p as absorption terms in [9] and as reaction terms in [11] on blow-up properties of solutions in a finite time. Naturally we want to know how diffusion coefficients u^m, v^n, reaction terms u^l, v^a, u^b, v^k and absorption terms $u^\alpha v^p, u^q v^\beta$ affect the blow-up properties of the system (1.1). We hope to give quantitative relations among these coefficients to decide whehter the global solution of (1.1) exists or the solution will blow up in a finite time. Our main result is: