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COMPARISON PRINCIPLE FOR VISCOSITY SOLUTIONS WITH HIGH GROWTH AT INFINITY

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Abstract This paper is concerned with the comparison principle for viscosity solutions of the nonlinear elliptic equation $F(Du, D^2u) + |u|^{s-1}u = f$ in $\mathbf{R}^N$, where f is uniformly continuous and F satisfies some conditions about p $(p > 2)$. We got the comparison principle for the viscosity solutions with some high growth at infinity, which relies on the relation between p and s.
Key Words Comparison principle; viscosity solution; high growth at infinity.
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1. Introduction

There are many papers concerning the equation

$$-\text{div}\,(|Du|^{p-2}Du) + |u|^{s-1}u = f(x), \quad x \in \mathbf{R}^N \tag{1}$$

In [1], H. Brezis studied the existence and uniqueness for the solutions when f hasn't any growth condition at infinity and $p = 2$, $s > 1$, (i.e. $-\Delta u + |u|^{s-1}u = f(x)$, $x \in \mathbf{R}^N$). In [2], T. Gallouët and J.M. Morel studied the case $p = 2$, $0 \le s \le 1$, they proved the existence and the characterization of minimal positive solution under some growth condition of f at infinity. Recently, L. Boccardo, T. Gallouët and J.L. Vazquez discussed the case $p > 2 - 1/N$, $s > p - 1$ in [3], they also proved the existence without any growth condition of f at infinity. All the solutions we mentioned above are due to Sobolev.

In this paper, we consider the fully nonlinear equations of the form

$$F(Du, D^2u) + |u|^{s-1}u = f(x), \quad x \in \mathbf{R}^N \tag{2}$$

where $s > 0$, F satisfies some conditions about p $(p > 2)$. Due to the fully nonlinearity, we discuss their viscosity solutions. There are also some papers about the viscosity solutions of the equations in this form. In [4], G. Diaz and R. Letelier extended the result of the case $p = 2$, $s > 1$ to fully nonlinear equations using viscosity theory. In [5] we discussed the existence of the viscosity solutions of (2).

The notation of viscosity solution was first introduced by P.L. Lions and M.G. Crandall in [6]. Viscosity theory is a new method for the study of fully nonlinear equations, for more it can be seen in [7], [8] etc. Comparison principle is the most important content in this theory. To get the comparison principle in $\mathbf{R}^N$, generally the viscosity solutions must be at most of linear growth at infinity such as in [7], [9], [10] etc. For the viscosity solutions with higher growth at infinity, [11] and [4] proved the comparison principle due to some special linearity of Δu. But their method fails in our case because of the fully nonlinearity, so we improved the method used in [7] and [9], so that it can deal with the solutions with higher growth. We consider only the nonnegative solutions and require some weaker conditions of F and some stronger conditions of f than those for the existence in [5].

All through the paper, we ask that $f \in UC(\mathbf{R}^N)$, $F \in C(\mathbf{R}^N \times \mathbf{S}^N)$ ($\mathbf{S}^N$ be the space of $N \times N$ real symmetric matrices), and F satisfies

(F1) $F(q, X) \le F(q, Y)$, for all $q \in \mathbf{R}^N$, $X, Y \in \mathbf{S}^N$, $X \ge Y$

(F2) $|F(q, X)| \le \Lambda |q|^{p-2} \|X\|$, $\exists$ constant $\Lambda > 0$, for all $q \in \mathbf{R}^N$, $X \in \mathbf{S}^N$

If we write the equation (1) in nondivergent form

$$-|Du|^{p-2} tr\,[I + (p-2) Du \otimes Du/|Du|^2] D^2 u + |u|^{s-1} u = f(x)$$

obviously

$$F(q, X) = -|q|^{p-2} tr\,[I + (p-2) q \otimes q/|q|^2] X$$

satisfies (F1), (F2).

First, we give a brief introduction to the viscosity theory.

Consider the equation

$$F(x, u, Du, D^2 u) = 0 \tag{3}$$

in $\Omega \subset \mathbf{R}^N$. We require that F satisfies two fundamental nontonicity conditions:

$$F(x, a, q, X) \le F(x, b, q, X)$$

for all $x \in \Omega$, $q \in \mathbf{R}^N$, $X \in \mathbf{S}^N$, $a, b \in \mathbf{R}$, $a \le b$, and

$$F(x, a, q, X) \le F(x, a, q, Y)$$

for all $x \in \Omega$, $a \in \mathbf{R}$, $q \in \mathbf{R}^N$, $X, Y \in \mathbf{S}^N$, $X \ge Y$.