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NUMERICAL ANALYSIS OF A FINITE ELEMENT, CRANK-NICOLSON DISCRETIZATION FOR MHD FLOWS AT SMALL MAGNETIC REYNOLDS NUMBERS

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Abstract. We consider the finite element method for time dependent MHD flow at small magnetic Reynolds number. We make a second (and common) simplification in the model by assuming the time scales of the electrical and magnetic components are such that the electrical field responds instantaneously to changes in the fluid motion. This report gives a comprehensive error analysis for both the semi-discrete and a fully-discrete approximation. Finally, the effectiveness of the method is illustrated in several numeral experiments.

Key words. Navier-Stokes, MHD, finite element, Crank-Nicolson

1. Introduction

Magnetohydrodynamics (MHD) is the theory of macroscopic interaction of electrically conducting fluid and electromagnetic fields. Many interesting MHD-flows involve a viscous, incompressible, electrically conducting fluid that interacts with an electromagnetic field. The governing equations for these MHD flows are the Navier-Stokes (NS) equations (NSE) coupled with the pre-Maxwell equations (via the Lorentz force and Ohm's Law). The resulting system of equations (see e.g. Chapter 2 in [21]) often requires an unrealistic amount of computing power and storage to properly resolve the flow details. A simplification of the usual MHD equations can be made by noting that most terrestrial applications involve small R_m; e.g. most industrial flows involving liquid metal have $R_m < 10^{-2}$. Moreover, it is customary to solve a quasi-static approximation when an external magnetic field is present R_m is small since the time scale of the fluid velocity is much shorter than that of the electromagnetic field [3]. We provide herein a stability and convergence analysis of a fully discrete finite element (FE) discretization for time-dependent MHD flow at a small Re_m and under a quasi-static approximation. Magnetic damping of jets, vortices, and turbulence are several applications, [3, 18, 20, 22].

Let Ω be an open, regular domain in $\mathbb{R}^d$ ($d = 2$ or 3). Let $R_m = UL/\eta > 0$ where U, L are the characteristic speed and length of the problem, $\eta > 0$ is the magnetic diffusivity. The dimensionless quasi-static MHD model is given by: Given time $T > 0$, body force $\mathbf{f}$, interaction parameter $N > 0$, Hartmann number $M > 0$, and domain $\Omega_T := (0, T] \times \Omega$, find velocity $\mathbf{u} : \Omega_T \to \mathbb{R}^d$, pressure $p : \Omega_T \to \mathbb{R}$, electric current density $\mathbf{j} : \Omega_T \to \mathbb{R}^d$, magnetic field $\mathbf{B} : \Omega_T \to \mathbb{R}^d$, and electric potential $\phi : \Omega_T \to \mathbb{R}$ satisfying:

$$
\begin{array}{rclrcl}
N^{-1}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u}) &=& \mathbf{f} + M^{-2}\Delta\mathbf{u} - \nabla p + \mathbf{j}\times\mathbf{B}, & \nabla\cdot\mathbf{u} &=& 0\\
-\nabla\phi + \mathbf{u}\times\mathbf{B} &=& \mathbf{j}, & \nabla\cdot\mathbf{j} &=& 0\\
\nabla\times\mathbf{B} &=& R_m\mathbf{j}, & \nabla\cdot\mathbf{B} &=& 0
\end{array} \tag{1}
$$

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subject to boundary and initial conditions

$$(2)\qquad \begin{array}{ll} \mathbf{u}(\mathbf{x},t)=0, & \forall(\mathbf{x},t)\in\partial\Omega\times(0,T] \\ \phi(\mathbf{x},t)=0, & \forall(\mathbf{x},t)\in\partial\Omega\times(0,T] \\ \mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x}), & \forall\mathbf{x}\in\Omega \end{array}$$

where $\mathbf{u}_0 \in V$ and $\nabla\cdot\mathbf{u}_0 = 0$. When $R_m << 1$, then $\mathbf{j}$ and $\nabla\times\mathbf{B}$ in (1)(3a) decouple. Suppose further that $\mathbf{B}$ is an applied (and known) magnetic field. Then (1) reduces to the simplified MHD (SMHD) system studied herein: Find $\mathbf{u}$, p, ϕ satisfying

$$(3)\qquad \begin{array}{rcl} N^{-1}(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}) &=& \mathbf{f}+M^{-2}\Delta\mathbf{u}-\nabla p+\mathbf{B}\times\nabla\phi+(\mathbf{u}\times\mathbf{B})\times\mathbf{B} \\ \nabla\cdot\mathbf{u} &=& 0 \\ -\Delta\phi+\nabla\cdot(\mathbf{u}\times\mathbf{B}) &=& 0. \end{array}$$

subject to (2). This is the time dependent version of the model first proposed by Peterson [19].

We provide a brief overview of previous applications and analyses of MHD flows (high and low R_m) in Section 1.1. In Section 2, we present notation and a weak formulation of (3) required in our stability and convergence analysis. In this report we prove stability estimates for any solution $\mathbf{u}$, p, ϕ to a semi-discrete and fully discrete approximation of (3) in Propositions 3.2, 4.2 respectively. We use these estimates to prove optimal error estimates in two steps:

- Semi-discrete (FE in space), Section 3
- Fully-discrete (FE in space, Crank-Nicolson time-stepping), Section 4

Let $h > 0$ and $\Delta t > 0$ be a representative measure of the spatial and time discretization. We investigate the interplay between spatial and time-stepping errors. We prove that the method is unconditionally stable and, for small enough Δt, the errors satisfy

$$\text{error}(\mathbf{u},p,\phi) < \mathcal{O}(h^r+\Delta t^2)\to 0, \quad \text{as } h,\Delta t\to 0$$

where r is the order of the FE approximation. See Theorems 3.3, 4.3 and Corollaries 3.4, 4.5.

1.1. Overview of MHD models. Applications of the MHD equations arise in astronomy and geophysics as well as numerous engineering problems including liquid metal cooling of nuclear reactors [2, 7], electromagnetic casting of metals [16], controlled thermonuclear fusion and plasma confinement [8, 23], climate change forecasting and sea water propulsion [15]. Theoretical analysis and mathematical modeling of the MHD equations can be found in [3, 10]. Existence of solutions to the continuous and a discrete MHD problem without conditions on the boundary data of $\mathbf{u}$ is derived in [24]. Existence and uniqueness of weak solutions to the equilibrium MHD equations is proven by Gunzburger, Meir, and Peterson in [6]. Meir and Schmidt provide an optimal convergence estimate of a FE discretization of the equilibrium MHD equations in [17]. To the best of our knowledge, the first rigorous numerical analysis of MHD problems was conducted by Peterson [19] by considering a small R_m, steady-state, incompressible, electrically conducting fluid flow subjected to an undisturbed external magnetic field. Further developments can be found in [1, 12, 13].