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REGULARITY RESULTS FOR A STRONGLY DEGENERATE PARABOLIC EQUATION

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Abstract M. Bertsch & R. Dal Passo proved the existence and uniqueness of the Cauchy problem for $u_t = (\varphi(u)\psi(u_x))_x$, where $\varphi > 0$, ψ is a strictly increasing function with $\lim_{s\to\infty} \psi(s) = \psi_\infty < \infty$. The regularity of the solution has been obtained under the condition $\varphi'' < 0$ or $\varphi = \text{const}$. In the present paper, under the condition $\varphi'' \le 0$, we give some regularity results. We show that the solution can be classical after a finite time. Further, under the condition $\varphi'' \le -\alpha_0$ (where α_0 is a constant), we prove the gradient of the solution converges to zero uniformly with respect to x as $t \to +\infty$.

Key Words Strongly degenerate parabolic equation; uniformly parabolic equation; supersolution.

Classification 35K.

1. Introduction

We consider the problem

$$\text{(I)} \qquad \begin{cases} u_t = (\varphi(u)\psi(u_x))_x & for\ x \in R,\ t \in (0,\infty) \\ u(x,0) = u_0(x) & for\ x \in R \end{cases}$$

where $\varphi : R \to R^+$ is smooth and positive, and $\psi : R \to R$ is a smooth, odd function such that $\psi' > 0$ in R and

$$\lim_{s\to\infty} \psi(s) = \psi_\infty \tag{1}$$

The initial function $u_0 : R \to R$ is bounded and strictly increasing. $\psi(u_0') \in C(R)$ (where $\psi(u_0'(x_0)) = \psi_\infty$ if u_0 is discontinuous at x_0).

In view of the condition (1), the function ψ' belongs to $L^1(R)$, ψ' is not uniformly bounded away from zero. Hence the equation (1) is of degenerate parabolic type. We call (1) is strongly degenerate parabolic type if the stronger condition (1) is satisfied.

M. Bertsch & R. Dal Passo solved the Cauchy problem (I) with the solution satisfying entropy condition[1]. Later, R. Dal passo[2] got the uniqueness of the solution.

They also gave some regularity results. They proved that if $\varphi'' \not\leq 0$ in R, then u is not necessarily continuous, even for $u_0 \in C^2(R)$. In the present paper, we prove that if $\varphi'' \leq 0$ and $u_0 \in C^2(R)$, then $u \in C^{2,1}(R \times R^+)$. For general $u_0(x)$, we show that the solution becomes classical after a finite time. i.e., if $\varphi'' \leq 0$ and $\psi'(s) \geq cs^{-2}$ as $s \to \infty$ (for some constant $c > 0$), then there exists a $T \geq 0$ such that u belongs to $C^{2,1}(R \times [T, \infty))$. Further, we discuss the asymptotic behaviour of the solution under the condition $\varphi'' \leq -\alpha_0$ (where α_0 is a constant) and prove the gradient of the solution converges to zero uniformly with respect to x as $t \to +\infty$.

The present paper is organized as follows. In Section 2, we list some related results up to date and our results. In Section 3, we prove the results.

2. Main Results

We first list the hypotheses and definitions:

H_1 $\psi \in C^2(R) \cap L^\infty(R)$, $0 < \psi' \leq c_1$ in R for some $c_1 > 0$, $\psi(0) = 0$, $\psi(-s) = -\psi(s)$, $\psi(s) \to \psi_\infty$ as $s \to \infty$.

H_2 $\varphi \in C^3(R)$, $\varphi > 0$ in R.

H_3 $u_0 \in L^\infty(R)$, u_0 is strictly increasing in R, $\psi(u_0') \in C(R)$, $\psi(u_0'(x)) \to 0$ as $x \to \pm\infty$.

Definition 1 *A function $u \in BV_{\text{loc}}(R \times [0, \infty)) \cap L^\infty(R \times R^+)$ is a solution of the problem* (I) *if*

(i) *For any $t \geq 0$, $u(.,t) \in BV_{\text{loc}}(R)$ and there exists a continuous function $\overline{\psi}$: $R \times [0, \infty) \to R$ such that*

$$\overline{\psi}(x,t) = \lim_{h \to 0} \psi\left(\frac{u(x+h,t) - u(x^-,t)}{h}\right)$$
$$= \lim_{h \to 0} \psi\left(\frac{u(x+h,t) - u(x^+,t)}{h}\right)$$

for any $x \in R$ and $t \geq 0$; here $u(x^\pm, t)$ respectively denotes one-sided limits.

(ii) *For any $\chi \in C^1(R \times [0, \infty))$ with compact support,*

$$\iint_{R \times R^+} (u\chi_t - \varphi(u)\overline{\psi}\chi_x)dxdt = -\int_R \chi(x,0)u_0(x)dx$$

The definition of the entropy condition is as follows.

Definition 2 *We say that a solution u of the problem* (I) *satisfies the entropy condition* (E) *if at any point $(x,t) \in R \times [0, \infty)$ in which u is discontinuous with respect to x, and*

$$\varphi(s) \leq \frac{\varphi(u^+) - \varphi(u^-)}{u^+ - u^-}(s - u^-) + \varphi(u^-) \tag{2}$$