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A STEFAN PROBLEM WITH CURVATURE CORRECTION IN A CONCENTRATED CAPACITY*

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Abstract We consider a Stefan problem with the curvature correction in a concentrated capacity. It is a kind of phase transition problems, in which the unknown temperature satisfies both the Stefan condition and the curvature correction on the free boundary, and also fulfills a kind of heat equations with special inner heat source. The inner heat source is related to the derivative of the temperature with respect to the direction vertical to the phase regions. The result established in this paper is the existence of a global weak solution of this problem.

Key Words Stefan problem; curvature; concentrated capacity.
Classification 35R35, 35B45.

1. Introduction

Consider a process of heat conduction in a complex material made by joining together two materials which have different critical temperatures of solidification. Let $x' = (x_1, \cdots, x_N) \in \mathbf{R}^N$ and $x = (x', x_{N+1}) \in \mathbf{R}^{N+1}$. Denote by Ω a bounded domain in $\mathbf{R}^N$ and by $Q_1 \equiv \Omega \times (0, b) \subset \mathbf{R}^{N+1}$ $(b > 0)$ and $Q_2 \equiv \Omega \times (-1, 0) \subset \mathbf{R}^{N+1}$ the regions occupied by the material A and the material B respectively. Suppose that the critical temperature of solidification of A is much greater than that of B, and the heat conduction in B in the direction of x_{N+1} is much greater than in other directions. Then, phase transitions will occur in B, not in A, when the temperature is greater than the critical point of solidification of B but less than that of A. The derivative of temperature with respect to x_{N+1} on the interface of A and B can be viewed as a heat source for the phase transition in B. Let u and w denote the temperature distribution in A and B respectively, then, if the influence of the shape of the free boundary in B upon phase transitions is neglected, this process can be modeled by a Stefan problem

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in a concentrated capacity as follows.

$$\begin{cases} u_t - \Delta u = f \quad \text{in } Q_{1T} \equiv Q_1 \times (0,T) & (1.1)\\ \dfrac{\partial u}{\partial x_{N+1}} = \psi \quad \text{on } \Omega(b) \times [0,T] & (1.2)\\ u = \varphi \quad \text{on } (\partial\Omega \times [0,b] \times [0,T]) \cup (\overline{Q}_1 \times \{0\}) & (1.3)\\ u = w \quad \text{on } \overline{\Omega(0)} \times [0,T] & (1.4)\\ \dfrac{\partial(w + \mathcal{H}(w))}{\partial t} - \Delta_{x'} w = g + \dfrac{\partial u}{\partial x_{N+1}} \quad \text{in } \mathcal{D}'(\Omega(0) \times (0,T)) & (1.5)\end{cases}$$

where $u \equiv u(x,t)$, $w \equiv w(x',t)$, $\Delta \equiv \Delta_x$, $\Omega(z) = \Omega \times \{z\}$, $(z = 0, b)$, $Q_1 \equiv \Omega \times (0,b)$.

Ardreucci, D. [1] and Magenes, E. [2] considered the system (1.1)–(1.5). They established the existence and uniqueness of a weak solution by different methods.

In this paper, we take into account the influence of the shape of the free boundary in B upon the phase transitions. In this setting, the equation (1.5) must be replaced by

$$\begin{cases} \dfrac{\partial(w + \chi)}{\partial t} - \Delta_{x'} w = g + \dfrac{\partial u}{\partial x_{N+1}} \quad \text{in } \mathcal{D}'(\Omega(0) \times (0,T)) & (1.6)\\ w = -\kappa \quad \text{on the free boundary} & (1.7)\\ \chi(0) = \chi_0 & (1.8)\end{cases}$$

where κ is the mean curvature of the free boundary with the convection that, when the solid region protrudes into the liquid region, κ is positive, and $\chi : \Omega \to \{0, 1\}$.

We shall use the idea in [3] to prove the existence of a weak solution of the system (1.1)–(1.4) and (1.6)–(1.8). Now we state our main results as follows.

Theorem 1.1 *Suppose that* $\Omega \subset \mathbf{R}^n$ *be a bounded Lipschitz region,* $\varphi \in C^1(\overline{Q}_1 \times [0,T])$, $f \in L^2(Q_1 \times (0,T))$, $g \in L^2(\Omega \times (0,T))$, $\psi \in L^2(\Omega \times (0,T))$, $\chi_0 \in BV(\Omega)$ *and* $\chi_0 : \Omega \to \{0,1\}$, *then the system* (1.1)–(1.4) *and* (1.6)–(1.8) *has at least one weak solution* $\{u, w, \chi\}$, $u - \varphi \in L^2(0,T; \widetilde{\mathcal{H}}_0^1(Q_1)) \cap L^\infty(0,T; L^2(Q_1))$, $w - \varphi \in L^2(0,T; \mathcal{H}_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$, $\chi \in L^\infty(0,T; BV(\Omega))$, *where* $\widetilde{\mathcal{H}}_0^1(Q_1) \equiv \{h(x) \in \mathcal{H}^1(Q_1) \mid h = 0$ *on* $\partial\Omega \times [0,b]\}$, *in the sense that*

$$\begin{aligned} &\int_0^T \int_{Q_1} [u\zeta_t - \nabla u \nabla \zeta + f\zeta] dx dt + \int_0^T \int_{\Omega(0)} [(\chi + w)\zeta_t - \nabla_{x'} w \nabla_{x'} \zeta + g\zeta] dx' dt \\ &+ \int_0^T \int_{\Omega(b)} \psi\zeta dx' dt + \int_{Q_1} \varphi(x,0)\zeta(x,0) dx \\ &+ \int_\Omega [\varphi(x',0,0) + \chi_0(x')]\zeta(x',0,0) dx' = 0, \\ &\forall \zeta \in C^1(\overline{Q}_1 \times [0,T]), \quad \zeta(x,T) = 0, \quad \zeta \mid_{\partial\Omega \times [0,b] \times [0,T]} = 0 \end{aligned} \tag{1.9}$$

and

$$\int_0^t \int_\Omega [\operatorname{div} \xi - \mu \cdot \nabla \xi \cdot \mu] |\nabla \chi| dt + \int_0^t \int_\Omega [\operatorname{div} (w\xi)] \chi dx' dt = 0,$$