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GLOBAL RESOLVABILITY FOR QUASILINEAR HYPERBOLIC SYSTEMS*

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Abstract In this paper, we consider the globally smooth solutions of diagonalizable systems consisted of n-equations. We give a sufficient condition which guarantees the global existence of smooth solutions. The techniques used in this paper can be applied to study the globally smooth (or continuous) solutions diagonalizable nonstrict hyperbolic conversation laws.
Key Words Global resolvability; maximum principle; function transformation.
Classification 35L65, 35L45, 35L60.

1. Introduction

It is well-known that the classical solutions of Cauchy problems for quasilinear hyperbolic systems, generally speaking, exist only locally in time and will occur singularities in finite time, even if the initial data are sufficiently smooth and small ([1]–[3]). However, there are certain examples of globally defined classical solutions ([4]). Hence it is of interesting to determine the conditions which guarantee the existence of globally classical solutions.

Under the case of diagonalizable 2×2 systems, systematic results have been obtained ([5]–[8]). The diagonalizable systems consisted of n-equations ($n > 2$) were studied first by D. Hoff ([9]). By using an inequality given by Rozdestvenskii in [10], paper [11] also gets the same results as that of [9] under less restrictions on the initial data. The method used in [9] and [11] require that the systems under consideration are strictly hyperbolic.

In this paper, we also consider the globally smooth solutions of diagonalizable systems consisted of n-equations. We give a sufficient condition (2.6) and two of its more applicable cases (2.31), (2.32) which guarantee the global existence of smooth solutions. In our analysis, we do not require the systems considered are strictly hyperbolic, and when the systems considered are strictly hyperbolic, the result in [9] or [11] is direct

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corollary of our results. Our analysis also indicates that our function transformation is a generalization of the function transformation proposed by P. Lax in [3].

The techniques used in this paper can also be applied to study the globally smooth (or continuous) solutions to diagonalizable nonstrict hyperbolic conservation laws. As an example, we consider the globally smooth (or continuous) solutions to isentropic gas dynamics systems in Euler coordinates.

2. An Existence Theorem of Globally Smooth Solutions for the Diagonalizable Systems of n-Equations

Consider the Cauchy problem

$$\begin{cases} \dfrac{\partial r^{(k)}}{\partial t} + \lambda^{(k)}(t,x,r)\dfrac{\partial r^{(k)}}{\partial x} = 0 \\ r^{(k)}\big|_{t=0} = r_0^{(k)}(x) \end{cases} \quad (k = 1, 2, \cdots, n) \tag{2.1}$$

where $r = \left(r^{(1)}, \cdots, r^{(n)}\right)$, $t \in R_+$, $x \in R$, n-positive integer, and

$$\Omega = \left\{(t,x,r)\big| |r^{(k)}| \le M,\ 0 \le t \le T,\ |x| < \infty,\ k = 1, 2, \cdots, n\right\} \tag{2.2}$$

Suppose that

(i) $r_0^{(k)}(x) \in C^1(R)$, and

$$\left\|r_0^{(k)}\right\|_{C^1(R)} \le M \tag{2.3}$$

(ii) $\lambda^{(k)}(t,x,r)$ are continuously differentiable functions on the domain Ω and

$$\left|\lambda^{(k)}\right| \le N,\ 0 \le \frac{\partial \lambda^{(k)}}{\partial x} \le N,\ \left|\frac{\partial \lambda^{(k)}}{\partial r^{(j)}}\right| \le N,\ j, k = 1, 2, \cdots, n \tag{2.4}$$

where M, N are positive constants, T is any given positive number.

We have the following theorem

Theorem 2.1 *Assume that the k-th characteristic field for the equation of the Cauchy problem* (2.1) *and corresponding initial data satisfy*

(iii) $\dfrac{\partial \lambda^{(k)}}{\partial r^{(k)}} \ge 0$, $\dfrac{dr_0^{(k)}(x)}{dx} \ge 0$, $k = 1, 2, \cdots, n$

and conditions (i) *and* (ii) *are satisfied.*

If there exist n functions $p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right)$ $(k = 1, 2, \cdots, n)$ *satisfying*

$$p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right) \in C^1(R^n),\ p^{(k)}\left(r^{(1)}, \cdots, r^{(n)}\right) > 0 \tag{2.5}$$

$$\frac{1}{n-1} p^{(k)} p^{(k)} \lambda_k^{(k)} + p^{(k)} p^{(j)} \lambda_j^{(j)} + p^{(j)} \left(p^{(k)}\left(\lambda^{(k)} - \lambda^{(j)}\right)\right)_{,j} \ge 0 \tag{2.6}$$

where $\lambda_j^{(k)} = \dfrac{\partial \lambda^{(k)}}{\partial r^{(j)}}$, $p_j^{(k)} = \dfrac{\partial p^{(k)}}{\partial r^{(j)}}$, $k, j = 1, 2, \cdots, n$,

then the Cauchy problem (2.1) admits a unique global smooth solution.