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ON THE OCCURRENCE OF "VACUUM STATES" FOR 2×2 QUASILINEAR HYPERBOLIC CONSERVATION LAWS*

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Abstract We show that the solution to the Cauchy problem of 2×2 nonlinear conservation laws, in general, may go out the strictly hyperbolic region of the system in a finite time, here the initial data are given in the strictly hyperbolic region. In other words, in general, we can't confine our attention to solve the Cauchy problem of 2×2 nonlinear conservation laws in strictly hyperbolic type. However, we can expect that it may be solved under the additional conditions (A) and (b).

Key Words "Vacuum states"; quasilinear hyperbolic conservation laws.
Classification 35L65.

1. Introduction

This paper is a development of the papers [1–5]. In [1–5], we considered the equations of isentropic gas dynamics in Lagrangian coordinates. We proved that the vacuum never occurs in an isentropic flow consisting of rarefaction waves even though the density may tend to zero as time tends to infinity. Now, we study the similar issue for a pair of quasilinear hyperbolic conservation laws.

The problem of the occurrence of the vacuum in isentropic flows has been a central issue in this field for some time, which was addressed by some authors, e.g. Liu and Smoller [6]. When the vacuum appears, the speeds of the characteristics of two families coincide with each other, the system is not strictly hyperbolic and waves behave in a singular way, causing serious analytical difficulty [7–9]. It is well known that the solution of Riemann problem for isentropic gas dynamics may contain the vacuum. One thereby tends to believe that if the initial data are not far from the vacuum, then it will occur at a later time. We showed that [1–3] the aforementioned solution of the Riemann problem indeed is the only circumstance where the vacuum can occur. Precisely, we showed that vacuum states cannot appear in rarefaction wave solutions of the equations unless the vacuum is present at time $t = 0^+$. It is well known that vacuum states can only appear due to the interaction of two rarefaction waves of different families, thus one tends to believe that they don't occur for the solutions in general. If it is proven to be true, then

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following DiPerna [10], we can get the existence theorem by compensated compactness theory, and avoid the serious analytical difficulties caused by the appearence of vacuum [7–9].

The similar issue is naturally addressed to a pair of quasilinear strictly hyperbolic conservation laws. This is a complicated problem. We can obtain a similar result under the additional assumptions (A) and (B) in Section 2. They hold for the equations of isentropic gas dynamics in Eulerian coordinates and that in Lagrangian coordinates. Once the assumption (A) or (B) is violated, no matter how narrowly, Examples 1 and 2 in Section 3 show that "vacuum states" may occur at a later time, where "vacuum state" means that, in the state, the speed of the characteristics of two families coincide with each other. But Example 3 shows that the condition (A) is not necessary for the nonoccurance of "vacuum".

Remark Examples 1 and 2 imply that the solution to the Cauchy problem of 2×2 nonlinear conservation laws, in general, may go out the strictly hyperbolic region of the system in a finite time, here the initial data are given in the strictly hyperbolic region and bounded away from the boundary. In other words, in general, we can't confine our attention to solve the Cauchy problem of 2×2 nonlinear conservation laws in the strictly hyperbolic region of the system. However, under the assumptions (A) and (B), we can expect the (generalized) solution will stay in the strictly hyperbolic region of the system in any finite time, if the initial data are given in it. In other words, we can expect that the Cauchy problem of 2×2 nonlinear conservation laws may be solved in the strictly hyperbolic region of the system under the assumptions (A) and (B).

Note The result in Section 2 is much more general than that in the papars [1–5], and the proof is much simpler [11–13]. It is owing to the intent of the papers [1–5] that is to find some suitable frameworks to study the solutions with shocks.

2. Existence Theorem

We are concerned with the Cauchy problem of a pair of conservation laws

$$u_t + f(u,v)_x = 0, \quad v_t + g(u,v)_x = 0, \quad (t,x) \in R_+ \times R \tag{E$_1$}$$

$$u(0,x) = u_0(x), \quad v(0,x) = v_0(x), \quad x \in R \tag{I$_1$}$$

here $f(u,v), g(u,v)$ are smooth in D_0, where D_0 is a region on (u,v) plane, the system $(\mathrm{E})_1$ is strictly hyperbolic in a subregion $D_1 \subset D_0$, i.e. the Jocobian matrix of the system $(\mathrm{E})_1$ has two real and distinct eigenvalues $\lambda < \mu$ in D_1. Let $G =: \mu - \lambda$, then $0 < G < \infty$ in D_1.

Riemann invariants of $(\mathrm{E})_1$ $z = z(u,v)$, $w = w(u,v)$ give a bijective smooth mapping from D_1 onto a region D on (z,w) plane. Thus $0 < G < \infty$ in D. Let $C_0 = \{(z,w) \mid G(z,w) = 0\}$, $C_\infty = \{(z,w) \mid G(z,w) = \infty\}$, we assume $\delta D = C_0 \cup C_\infty$, where δD denotes the boundary of D. The Riemann invariants diagonalize the principal part of the system $(\mathrm{E})_1$ as

$$z_t + \lambda(z,w)z_x = 0, \quad w_t + \mu(z,w)w_x = 0, \quad (t,x) \in R_+ \times R \tag{E}$$