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DETERMINATION OF THE UNKNOWN BOUNDARY CONDITIONS IN A TWO-PHASE STEFAN PROBLEM

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Abstract A two-phase Stefan problem with the heat flux boundary conditions, including an unknown function f, is considered. The existence, uniqueness, and continuous dependence upon the initial data of the solution (f, s, u_1, u_2) are proved.
Key Words Unknown boundary conditions; two-phase Stefan problem.
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1. Introduction

Consider the following two-phase Stefan problem

$$u_{1,t} - a_1^2 u_{1,xx} = 0, \qquad (x,t) \in Q_{1,T} \tag{1.1}$$
$$u_{2,t} - a_2^2 u_{2,xx} = 0, \qquad (x,t) \in Q_{2,T} \tag{1.2}$$
$$u_1(x,0) = \varphi_1(x), \qquad 0 \le x \le s(0) = b \tag{1.3}$$
$$u_2(x,0) = \varphi_2(x), \qquad b \le x \le 1 \tag{1.4}$$
$$a_1^2 u_{1,x}(0,t) = -f(u_1(0,t)), \qquad 0 < t \le T \tag{1.5}$$
$$a_2^2 u_{2,x}(1,t) = f(u_2^2(1,t)), \qquad 0 < t \le T \tag{1.6}$$
$$u_1(s(t),t) = u_2(s(t),t) = 0, \qquad 0 < t \le T \tag{1.7}$$
$$\dot{s}(t) = -k_1 u_{1,x}(s(t),t) + k_2 u_{2,x}(s(t),t), \qquad 0 < t \le T \tag{1.8}$$
$$u_1(0,t) = h(t), \qquad 0 < t \le T \tag{1.9}$$

where $Q_{1,T} = \{(x,t) \mid 0 < x < s(t), 0 < t \le T\}$, $Q_{2,T} = \{(x,t) \mid s(t) < x < 1, 0 < t \le T\}$. The constants $a_1 > 0$, $a_2 > 0$, $k_1 > 0$, $k_2 > 0$, $T > 0$, $0 < b < 1$, and the functions $\varphi_1(x), \varphi_2(x), h(t)$ are given, the function f included in the heat flux boundary conditions (1.5) and (1.6), and the functions $s(t), u_1(x,t), u_2(x,t)$ are unknown.

We first set the following definition.

Definition 1.1 *A set of functions (f, s, u_1, u_2) is called a solution of the problem (1.1)–(1.9) in $[0,T]$, if*

$f(\theta) \in C^{\frac{1}{2}}(I)$, where I is the range of $h(t)$ for $0 \le t \le T$;

$s(t) \in C^1([0,T])$, $0 < s(t) < 1$ *for* $0 \leq t \leq T$;
$u_i(x,t)$ *and* $u_{i,x}(x,t) \in C(\overline{Q}_{i,T})$, $i = 1, 2$;
$u_{i,t}(x,t)$ *and* $u_{i,xx}(x,t) \in C(Q_{i,T})$, $i = 1, 2$;
(1.1)–(1.9) *are satisfied.*

A similar problem has been considered in the parabolic initial-boundary value problem in [2]. But in order to apply the iteration method to the integral equation which the unknown f satisfies, there are not only the assumptions on the data, but also an additional assumption: for each t, $0 \leq t \leq T$, the function $u(1,t,f)$ lies in the interval $[h(0), h(t)]$.

It is the purpose of this paper to show that the problem (1.1)–(1.9) has a unique solution (f, s, u_1, u_2) in a small interval of time t under the assumptions

(H_1) $\varphi_1(x) \in C^2([0,b])$, $\dot{\varphi}_1(0) = \varphi_1(0) = \varphi_2(b) = 0$;

(H_2) $\varphi_2(x) \in C^2([b,1])$, $\dot{\varphi}_2(1) = \varphi_2(1) = \varphi_2(b) = 0$;

(H_3) $h(0) = 0$, $H_1 < \dot{h}(t) \leq H_2$ for $0 \leq t \leq T$, $|\dot{h}(t_2) - \dot{h}(t_1)| \leq H_3(t_2 - t_1)$ for $0 \leq t_1 \leq t_2 \leq T$, where the constants $H_2 \geq H_1 \geq 16a_2^2\pi^{-1}(4\|\dot{\varphi}_1\|_{C([0,b])} + 5\|\dot{\varphi}_2\|_{C([b,1])} + 1)^2$, and $H_3 \geq 0$.

However, one has no need for the assumption on the unknown $u_2(1,t)$.

This paper is organized as follows. In Section 2, we introduce a sequence of approximate solutions and investigate the properties of this sequence. The existence of solutions is proved by contraction mapping in Section 3. The uniqueness and continuous dependence upon the initial data are discussed in the last section.

2. Sequence of Approximate Solutions

Now let (f, s, u_1, u_2) be a solution of (1.1)–(1.9). In such a case, the function f included in (1.5) and (1.6) has been given, and (s, u_1, u_2) solves the two-phase Stefan problem (1.1)–(1.4), (1.6)–(1.9). As is now well known, the solution (s, u_1, u_2) of (1.1)–(1.4), (1.6)–(1.9) is given by

$$u_1(x,t) = a_1^2 \int_0^t h(\tau) G_{1,\xi}(x,t,0,\tau)d\tau + a_1^2 \int_0^t v_1(\tau) G_1(x,t,s(\tau),\tau)d\tau + \int_0^b \varphi_1(\xi) G_1(x,t,\xi,0)d\xi \tag{2.1}$$

$$u_2(x,t) = \int_0^t f(u_2^2(1,\tau)) N_2(x,t,1,\tau)d\tau - a_2^2 \int_0^t v_2(\tau) N_2(x,t,s(\tau),\tau)d\tau + \int_b^1 \varphi_2(\xi) N_2(x,t,\xi,0)d\xi \tag{2.2}$$

$$s(t) = b - \int_0^t (k_1 v_1(\tau) - k_2 v_2(\tau))d\tau \tag{2.3}$$

where

$$(v_1(t), v_2(t)) = (u_{1,x}(s(t),t), u_{2,x}(s(t),t))$$