J.Partial Diff. Eqs. 8(1995), 261–272

Vol.8, No.3

POSITIVE SOLUTION OF A SEMILINEAR ELLIPTIC EQUATION ON R^N*

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(Received June 24, 1993; revised Apr. 28, 1994)

Abstract In this paper, we obtain the existence of positive solution of

$$\begin{cases} -\Delta u = b(x)(u-\lambda)_+^p, & x \in R^N \\ \lambda > 0, |\nabla u| \in L^2(R^N), & u \in L^{\frac{2N}{N-2}}(R^N) \end{cases}$$

under the assumptions that $1 < p < \dfrac{N+2}{N-2}$, $N \geq 3$, $b(x)$ satisfies

$$b(x) \in C(R^N), b(x) > 0 \text{ in } R^N$$

$$b(x) \xrightarrow[|x|\to\infty]{} b^\infty \text{ and } b(x) > \frac{4}{p+3} b^\infty \text{ for } x \in R^N$$

Key Words Elliptic equations; positive solution; critical point.
Classification 35J20, 35J60.

1. Introduction

In this paper, we consider the existence of solutions for the elliptic equation

$$\begin{cases} -\Delta u = b(x)(u-\lambda)_+^p \text{ in } R^N \\ u > 0 \text{ in } R^N, |\nabla u| \in L^2(R^N), \ u \in L^{\frac{2N}{N-2}}(R^N) \end{cases} \tag{1.1}$$

where $\lambda > 0$, $1 < p < \dfrac{N+2}{N-2}$, $N \geq 3$ and $b(x) \in C(R^N)$ is a bounded positive function, $(u-\lambda)_+ = \max\{u-\lambda, 0\}$.

When $b(x)$ is radial, P.L. Lions [1] has studied (1.1). A. Bahri and P.L. Lions [2] have obtained the existence of a positive solution under the conditions that

$$\begin{cases} b(x) \to b^\infty \text{ as } |x| \to +\infty \\ b(x) \geq b^\infty - C|x|^{2-N} \text{ for } |x| \geq R_0 \end{cases} \tag{1.2}$$

where C and R_0 are some positive constants.

* This work supported partially by Youth Foundation, NSFC and C.G. Project of Wuhan.

In this paper we prove

Theorem 1.1 *Suppose that $b(x)$ satisfies*

(H) $$b(x) \xrightarrow[|x|\to\infty]{} b^\infty, b(x) \geq \frac{4}{p+3} b^\infty \text{ for all } x \in \boldsymbol{R}^N$$

then (1.1) *admits a solution.*

Since $b^\infty = 0$ has been discussed in [2], hereafter, we will always assume $b^\infty > 0$.

2. Auxiliary Results

Let H denote the closure of $C_0^\infty(\boldsymbol{R}^N)$ under the norm $\|u\| = \left(\int_{\boldsymbol{R}^N} |\nabla u|^2 dx\right)^{\frac{1}{2}}$, we know that for $u \in H$, $\int_{\boldsymbol{R}^N} |u|^{\frac{2N}{N-2}} dx < +\infty$. Denote the best Sobolev constant by S, that is

$$S = \inf \left\{ \int_{\boldsymbol{R}^N} |\nabla u|^2 dx \Big| u \in H, \int_{\boldsymbol{R}^N} |u|^{\frac{2N}{N-2}} dx = 1 \right\}$$

For any $u \in H$, $\varphi \in H$ we have

$$\int_{\boldsymbol{R}^N} (u-\lambda)_+^p \varphi dx \leq \lambda^{p-\frac{N+2}{N-2}} S^{-\frac{N}{N-2}} \|u\|^{\frac{N+2}{N-2}} \|\varphi\| \tag{2.1}$$

To prove (2.1), let us notice that for $u \in H$, from $\left(\int_{\boldsymbol{R}^N} |u|^{\frac{2N}{N-2}} dx\right)^{\frac{N-2}{2N}} \leq S^{-\frac{1}{2}} \|u\|$ we get

$$\text{mes}\{x \in \boldsymbol{R}^N \mid u(x) \geq \lambda\} \leq S^{-\frac{N}{N-2}} \lambda^{\frac{-2N}{N-2}} \|u\|^{\frac{2N}{N-2}} \tag{2.2}$$

(2.1) follows from Hölder's inequality and (2.2).

The variational functional of (1.1) is

$$I(u) = \frac{1}{2} \int_{\boldsymbol{R}^N} |\nabla u|^2 dx - \frac{1}{p+1} \int_{\boldsymbol{R}^N} b(x)(u-\lambda)_+^{p+1} dx \tag{2.3}$$

From (2.1), $I(u)$ is well defined and continuously differentiable.

By [1, 2], the equation

$$\begin{cases} -\Delta w = b^\infty (w-\lambda)_+^p \text{ in } \boldsymbol{R}^N \\ w > 0 \text{ in } \boldsymbol{R}^N, \ w \in H \end{cases} \tag{2.4}$$

has a unique positive solution ω up to a translation and for some constants $C_0 > 0$, $C_1 > 0$, ω satisfies

$$\begin{cases} \omega(x)|x|^{N-2} \xrightarrow[|x|\to\infty]{} C_0 \\ |\nabla \omega(x)||x|^{N-1} \xrightarrow[|x|\to\infty]{} (N-2)C_0 \end{cases} \tag{2.5}$$

$$|D^2\omega(x)||x|^N \leq C_1, \text{ for } |x| \geq 1 \tag{2.6}$$