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ON THE CAHN-HILLIARD EQUATION WITH NONLINEAR PRINCIPAL PART*

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Abstract We study the Cahn-Hilliard equation with nonlinear principal part

$$\frac{\partial u}{\partial t}+D[m(u)(kD^3u-DA(u))]=0$$

The existence of classical solutions is established by means of the method based on Campanato spaces and the energy estimates. The corresponding uniqueness is also proved.

Key Words Cahn-Hilliard equation; existence; uniqueness; Campanato space.
Classification 35K.

1. Introduction

The Cahn-Hilliard equation, namely

$$\frac{\partial u}{\partial t}+D[m(u)(kD^3u-DA(u))]=0 \quad \text{in } Q_T=(0,T)\times(0,1) \tag{1.1}$$

is based on a continuum model for phase transition in binary system such as alloy, glasses and polymer-mixtures, see [1], [2]. Here $u(t,x)$ is the concentration of one of the phase of the system, $m(u)$ the mobility, k a positive constant,

$$J=m(u)(kD^3u-DA(u))$$

the net flux, $D=\frac{\partial}{\partial x}$. Based on physical consideration, the equation (1.1) is supplemented with the zero net flux boundary value condition

$$J\Big|_{x=0,1}=0 \tag{1.2}$$

the natural boundary value condition

$$Du\Big|_{x=0,1}=0 \tag{1.3}$$

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and the initial value condition

$$u(0,x) = u_0(x) \tag{1.4}$$

Using the energy method, Elliott and Zheng Songmu [3] have successfully treated the problem (1.1)–(1.4) for (1.1) with linearized principal part, namely for (1.1) in which $m(u)$ is a positive constant. And for $A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3$ with $\gamma_2 > 0$, the global existence of classical solutions of the problem (1.1)–(1.4) is established.

Since, in many situations, the mobility $m(u)$ depends on the concentration u in general, the investigation for (1.1) with nonlinear principal part seems to be a natural continuation of the pioneering work [3]. In this paper, we discuss the solvability of classical solutions of the problem (1.1)–(1.4) under the following much more general assumptions

$$m(s) > 0, \quad H(s) \equiv \int_0^s A(\sigma)d\sigma \geq -\mu, \quad \mu > 0$$

in which the non-uniform parabolicity for (1.1) is allowed and $A(s)$ is permitted to be some polynomial of odd order like $-s + \gamma_1 s^2 + \gamma_2 s^3$ with $\gamma_2 > 0$. The discussion for the degenerate case of the equation (1.1) will be subsequently presented in our next paper, where certain structure conditions are proposed ensuring the existence of "Physical Solutions", namely the solutions with the property that $0 \leq u \leq 1$. An interesting work for such kind of equation of degenerate type can be found in the recent work by Bernis and Friedman [7].

The main difficulties for treating the problem (1.1)–(1.4) are caused by the nonlinearity of the principal part and the lack of maximum principle. Due to the nonlinearity of the principal part, there are more difficulties in establishing the global existence of classical solutions. The method we use is based on the Schauder type priori estimates, which are relatively less used for such kind of parabolic equations of fourth order. Here the Schauder type estimates will be obtained by means of a modified Campanato space. We note that the Campanato spaces have been widely used to the discussion of partial regularity of solutions of parabolic systems of second order. Because of the lack of maximum principle, the actually used Campanato space is a modified version. In fact, after such modification, the terms related to supremum norm will not appear in deriving the key estimate (3.9). A detailed description and the associated properties of such space will be given in Section 2. Subsequently we prove the existence of classical solutions of the problem (1.1)–(1.4). The uniqueness is also discussed in this section.

2. A Modified Campanato Space and the Hölder Norm Priori Estimates

Let $Q_T = (0,T) \times (0,1)$, $y_0 = (t_0, x_0) \in \overline{Q}_T$. For any fixed $R > 0$, define