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THE ESTIMATES NEAR THE BOUNDARY FOR SOLUTIONS OF MONGE-AMPERE EQUATIONS

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Abstract The present paper is concerned with the global C^2-estimates of solutions to boundary value problems for degenerate elliptic Monge-Ampere equations. Both of degeneracies of the boundary data and the right hand side of the equation are considered. In two dimensional case a result about smooth solutions is also contained.

Key Words Monge-Ampere equation, degenerate elliptic, non constant Dirichlet data.

Classification 35J60, 35J70

1. Introduction

The present paper is devoted to Dirichlet problems for degenerate elliptic Monge-Ampere equations of the form

$$\det(D^2u) = K(x)f(x, u, Du) \quad \text{in } \Omega \tag{1.1}$$

with nonhomogeneous boundary data

$$u = \phi \quad \text{on } \partial\Omega \tag{1.2}$$

We always assume that

$$\begin{gathered}\Omega \text{ is a smooth bounded convex domain} \\ \phi \text{ is smooth on } \partial\Omega \text{ and } K \geq 0,\ f > 0 \text{ in } \bar{\Omega} \times R^1 \times R^n\end{gathered} \tag{1.3}$$

In addition, K and f are C^2-continuous in the closure of the domain under consideration and subject to the following structure conditions:

$$f_u(x, u, p) \geq 0 \quad \text{on } \bar{\Omega} \times R^1 \times R^n \tag{1.4}$$

$$K(x) \leq \mu(\text{dis}(x, \partial\Omega))^\beta, \quad f(x, \phi(x), p) \leq \mu(1 + p^2)^{\alpha/2} \tag{1.5}$$

for all x near $\partial\Omega$ and $p \in R^n$, and

$$f(x, -N, p) \leq 1/h(p), \quad x \in \bar{\Omega}, p \in R^n \tag{1.6}$$

where N, α, β, μ are nonnegative constants, $\beta \geq \alpha - n - 1$ and h is a positive function in $L^1_{\text{loc}}(R^n)$ such that

$$\int_\Omega K dx < \int_{R_n} h dx \tag{1.7}$$

There have been many papers to devoted to the problem (1.1) (1.2). For example, [1] for $f = 1$ and [2] for $K^{1/n} \in C^{1,1}(\bar{\Omega})$ respectively obtained solutions in $C^{1,1}(\Omega) \cap C^{0,1}(\bar{\Omega})$; [3] for ϕ=constant and [4][5] for general boundary data under some mild restrictions, proved the solvability in $C^{1,1}(\bar{\Omega})$. It seems that there is a big gap between the results about constant data and general data. Bedford and Fornaess [6] pointed out, generally speaking, the solution for the degenerate case is of the class in $C^{1,1}(\bar{\Omega})$ at most. Recently, [7] [8] studied the solvability in $C^\infty(\bar{\Omega})$ for two dimensional case if ϕ=constant and $K(x)$ clearly changes its sign on $\partial\Omega$, i.e.

$$K > 0 \text{ in } \Omega, \quad K(x) = 0 \neq dK \text{ on } \partial\Omega \tag{1.8}$$

The purpose of this paper is to extend the result in [5] to the case where $\Omega \subset R^n$ and the boundary data are of higher degree degeneracy and both degeneracies of ϕ and $K(x)$ are considered in the meantime. For each $p \in \partial\Omega$, by the convexity of Ω, we can locally express $\partial\Omega$ as

$$x_n = g(x'), \quad \text{where } g(0) = \partial_j g(0) = 0, \quad j = 1, \cdots, n-1 \\ \text{and } (g_{ij}(0)) \text{ positive} \tag{1.9}$$

Later, for simplicity we always call such kinds of coordinate systems like (1.9) the standard coordinates at p. Let $S_0 = \{x \in \partial\Omega | K(x) = 0\}$. We denote by Σ the set of all the points $p \in S_0$ such that under (1.9) for each $k = 1, \cdots, n-1$, there is an integer $N = N(k) \geq 2$ and a vector $\xi \in R^{n-1} \setminus \{0\}$ satisfying

$$\sum_{|\alpha|=N'} \xi^\alpha (g_{kk}\partial^\alpha \phi - \partial^\alpha g \phi_{kk})(0) = 0 \quad \text{if} \quad 2 \leq N' \leq N-1 \tag{1.10}$$

and

$$\sum_{|\alpha|=N} \xi^\alpha (g_{kk}\partial^\alpha \phi - \partial^\alpha g \phi_{kk})(0) \neq 0 \quad \text{if } N \text{ is odd} \\ \text{or negative if } N \text{ is even} \tag{1.11}$$

The motivation of introducing this definition is due to C.Zuily (by private communication). Checking the definition of D_0 in [5], one can easily find that Σ contains D_0. In fact, the degenerate points studied in [5], only correspond to (1.11) where $N = 2$ or $N = 3$ whereas the present case contains more boundary points of higher degree degeneracy. A uniformly C^2-bounds on $\partial\Omega$ for points in Σ can be obtained although K vanishes there.

To describe the influence of both degeneracies of the boundary data and the right hand side of the equation, we introduce another kind degenerate points of the boundary