J.Partial Diff. Eqs., 7 (1994), 193–214
©International Academic Publishers Vol.7, No.3

A FREE BOUNDARY PROBLEM ARISING IN SMOULDER COMBUSTION*

Bei Hu
(Department of Mathematics, University of Notre Dame,
Notre Dame, Indiana 46556, USA.)
(Received Apr. 8, 1992; revised June 22, 1993)

Abstract The smouldering combustion is modeled as a free boundary problem here. By using the Duvaut's transform, the problem is reduced to a variational inequality. Existence and uniqueness are established. The properties of the free boundary are studied in various cases. The asymptotic behavior of the free boundary with respect to the parameter is rigorously proved, which confirms the result of a previous work by J. Adler and D. M. Herbert, obtained by using asymptotic expansion. Furthermore, we show that the time dependent problem will actually converge to a travelling wave solution if the boundary data converge to the corresponding travelling wave solution.

Key Words Free boundary, *a priori* estimate, variational inequality, asymptotic behavior.

Classifications 35B40, 35R35.

1. Introduction

There are many flame propagation models, see [1, 2, 3] and the references therein. The smouldering combustion is a slow burning process such as burning of a paper.

Assume that the thickness of the reaction zone is negligible, a two dimensional flame propagation model is derived in [1] as follows:

$$\frac{\partial u}{\partial t} = \varepsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \quad \text{for } y < f(x,t), x > 0, t > 0 \tag{1.1}$$

with the boundary conditions

$$u = 1 \quad \text{on } y = 0, x > s(t) \tag{1.2}$$

$$u = 0 \quad \text{on } y = f(x,t) \tag{1.3}$$

* The project partially supported by USA National Science Foundation Grant DMS–9024986.

and the free boundary conditions:

$$f(s(t),t) = 0 \tag{1.4}$$

$$-\frac{\partial}{\partial x}f(x,t) = \frac{\partial}{\partial y}u(x,f(x,t)) - \varepsilon\frac{\partial}{\partial x}u(x,f(x,t))\frac{\partial}{\partial x}f(x,t) \tag{1.5}$$

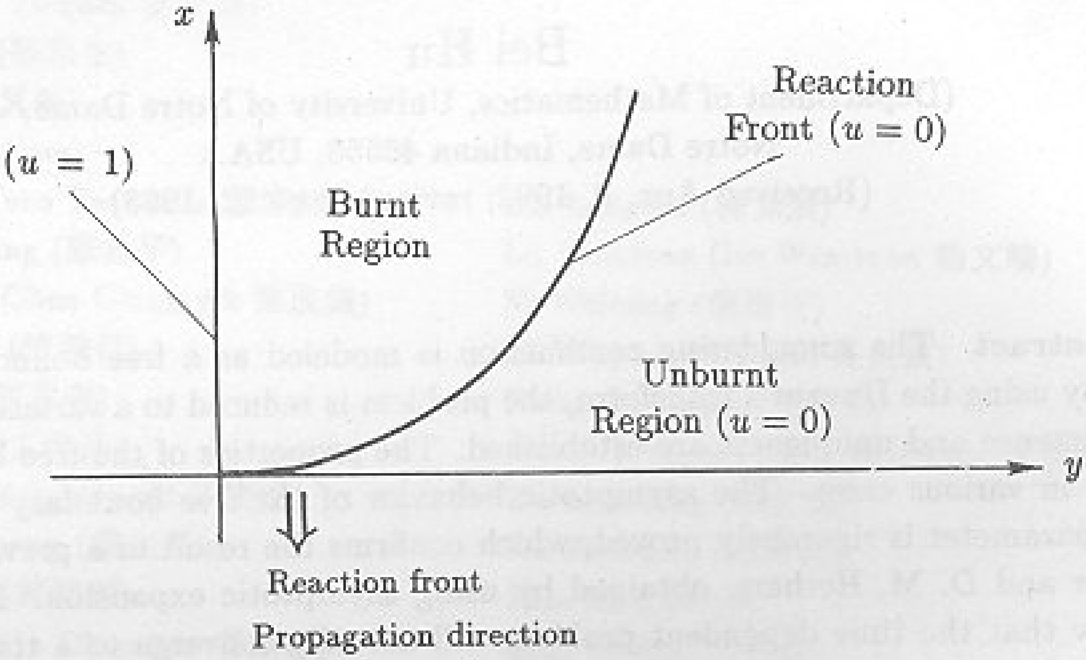

Figure 1.

where u is the oxidizer concentration and $y = f(x,t)$ is the reaction front and $s(t)$ is the reaction front on the x-axis (as in Figure 1); $\sqrt{\varepsilon}$ is proportional to ρ_0/ρ_f, ρ_0 is the mean density of the oxidizer, and ρ_f is the density of the solid reactant. All the quantities have been nondimensionalized and $\sqrt{\varepsilon}$ is typically small.

In Section 2, we study the travelling wave solutions. The Duvaut's transform reduces the problem to a variational inequality, and existence and uniqueness are therefore obtained. For this variational inequality, however, the solution will not solve the original problem (1.1)–(1.5), which will be shown in Section 3. In fact, the system (1.1)–(1.5) has no solutions if $\varepsilon > 0$; this is due to the assumption that the oxidizer concentration is discontinuous at the reaction front on the x-axis. After modifying the assumption so that the oxidizer at the reaction front is continuous, we obtain travelling wave solutions. In Section 3, we also establish the monotonicity of the free boundary. In Section 4, we show that the free boundary will converge to the corresponding free boundary of the Stefan problem as $\varepsilon \to 0$.

Finally in Section 6, we study the time dependent problem and show that the solution will converge to the travelling wave solution as $t \to \infty$ under certain assumptions.

2. Travelling Wave Solutions

We look for travelling wave solutions, namely, the solution which is a function of the variable $(x+t,y)$ only. As in [1], by making a change of variable $x+t=\xi$, letting