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$C^{1,\alpha}$-PARTIAL REGULARITY OF NONLINEAR PARABOLIC SYSTEMS*

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Abstract We prove $C^{1,\alpha}$-partial regularity of weak solution of nonlinear parabolic systems

$$u_t^i - D_\alpha A_i^\alpha(x,t,u,Du) = B_i(x,t,u,Du), \quad i = 1,\cdots,N$$

under the main assumption that A_i^α and B_i satisfy the natural growth condition.
Key Words Nonlinear parabolic system; partial regularity; natural growth condition
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1. Introduction

In this paper we will extend some of the partial regularity results for nonlinear elliptic systems to parabolic case. Actually, we intend to show that the method developed in [1], [3] can be also used to study nonlinear parabolic systems.

Let Ω be an open set in R^n. $T > 0$ and $Q = \Omega \times [0,T]$, and let $z = (x,t)$, where $x \in \Omega$, $0 < t \leq T$, denote a point in Q and $\partial_p Q$ the parabolic boundary of Q. Let $u(z) = (u^1(z),\cdots,u^N(z))$ be a vector valued function defined in Q. Denote by Du the gradient of u, i.e., $Du = \{D_\alpha u^i\}_{i=1,\cdots,N;\alpha=1,\cdots,n}$.

Consider the nonlinear parabolic systems of the following type

$$u_t^i - D_\alpha A_i^\alpha(z,u,Du) = B_i(z,u,Du), \quad i = 1,\cdots,N \tag{1.1}$$

We suppose that A_i^α and B_i satisfy the natural growth condition:

$$A_i^\alpha(z,u,p)p_\alpha^i \geq \lambda|p|^2 - f^2, \quad f \in L^\sigma(Q) \tag{1.2}$$

$$|A_i^\alpha(z,u,p)| \leq C(|p| + f_i^\alpha), \quad f_i^\alpha \in L^\sigma(Q) \tag{1.3}$$

$$|B_i(z,u,p)| \leq a(|p|^2 + f_0), \quad f_0 \in L^\tau(Q) \tag{1.4}$$

or

$$|B_i(z,u,p)| \leq C(|p|^{2-\delta} + f_i), \quad f_i \in L^\tau(Q) \tag{1.4$'$}$$

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where $\lambda > 0$, $a \geq 0$ and δ are constants with $0 < \delta < \dfrac{n}{n+2}$. We denote

$$V_N(Q) = L^2(0,T;H^1(\Omega,R^N)) \cap L^\infty(Q,R^N)$$

$$W(Q) = L^2(0,T;H_0^1(\Omega,R^N)) \cap H^1(0,T;L^2(Q,R^N))$$

By a weak solution of (1.1) under the natural growth condition (1.2)–(1.4) (or (1.2), (1.3), (1.4)$'$) we mean a vector valued function $u \in V_N(Q)$ such that

$$\int_Q [A_i^\alpha(z,u,Du)D_\alpha\varphi^i - u^i\varphi_t^i]dz = \int_Q B_i(z,u,Du)\varphi^i dz \tag{1.1$'$}$$

for all $\varphi \in W(Q) \cap L^\infty(Q,R^N)$ with $\varphi(x,0) = 0$, $\varphi(x,T) = 0$, $\forall x \in \Omega$.

For $z_0 = (x_0,t_0) \in Q$, denote

$$B_R = B(x_0,R) = \{x \in R^n, |x - x_0| < R\}$$

$$I_R = I(t_0,R) = \{t \in R, t_0 - R^2 < t < t_0\}$$

$$Q_R = Q(z_0,R) = B(x_0,R) \times I(t_0,R)$$

We prove the main theorem:

Theorem 1.1 *Let $u \in V_N(Q)$ be a weak solution of system* (1.1). *Suppose that A_i^α and B_i satisfy*

(H$_1$) $|A_i^\alpha(z,u,p)| \leq C(|p|+1)$

(H$_2$) $\dfrac{\partial A_i^\alpha(z,u,p)}{\partial p_\beta^j}\xi_\alpha^i\xi_\beta^j \geq \lambda|\xi|^2, \quad \lambda > 0, \quad \forall \xi \in R^{nN}$

(H$_3$) $A_i^\alpha(z,u,p)(i = 1,\cdots,N; \alpha = 1,\cdots,n)$ *are of class C^1 with bounded continuous derivative*

$$\left|\frac{\partial A_i^\alpha}{\partial p_\beta^j}\right| \leq L$$

(H$_4$) $(1+|p|)^{-1}A_i^\alpha(z,u,p)$ *are Hölder continuous in (z,u) uniformly with respect to p, i.e.,*

$$|A_i^\alpha(z,u,p) - A_i^\alpha(y,v,p)| \leq c(1+|p|)\eta(|u|,|z-y|^2 + |u-v|^2)$$

where $\eta(s_1,s_2) \leq K(s_1)\min(s_2^{\gamma/2},L)$ for some γ, $0 < \gamma < 1$ and $L > 0$, $K(t)$ is an increasing function,

(H$_5$) $|B_i(z,u,p)| \leq a(|p|^2 + b)$, $2aM < \lambda$, $\sup\limits_Q |u| = M$

or

$$|B_i(z,u,p)| \leq C(|p|^{2-\delta} + b)$$

Then the first derivatives $D_\alpha u^i$ of u are Hölder continuous in an open set $Q_0 \subset Q$ with meas $(Q \setminus Q_0) = 0$.

In proving the theorem stated above, we need the following Lemma which can be found in [5].