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AN OPTIMAL 9-POINT FINITE DIFFERENCE SCHEME FOR THE HELMHOLTZ EQUATION WITH PML

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Abstract. In this paper, we analyze the defect of the rotated 9-point finite difference scheme, and present an optimal 9-point finite difference scheme for the Helmholtz equation with perfectly matched layer (PML) in two dimensional domain. For this method, we give an error analysis for the numerical wavenumber's approximation of the exact wavenumber. Moreover, based on minimizing the numerical dispersion, we propose global and refined choice strategies for choosing optimal parameters of the 9-point finite difference scheme. Numerical experiments are given to illustrate the improvement of the accuracy and the reduction of the numerical dispersion.

Key words. Helmholtz equation, PML, 9-point finite difference scheme, numerical dispersion.

1. Introduction

The Helmholtz equation

$$-\Delta u - k^2 u = f, \tag{1.1}$$

governs wave propagations and scattering phenomena arising in many areas, for example, in aeronautics, marine technology, geophysics and optical problems. In practice, wave equation modeling in the frequency domain has many advantages over time domain modeling. For example, for certain geometries, only a few frequency components are required to perform wave equation inversion and tomography. Moreover, each frequency can be computed independently, which favors parallel computing. Multiexperiment seismic data can also be simulated economically once the impedance matrix is factored. In addition, modeling the effects of attenuation is more flexible in the frequency domain than in the time domain, because in the frequency domain we can directly input the attenuation coefficient as a function of frequency.

To compute the solution of the above problem, due to finite memory of the computer, absorbing boundary conditions are needed to truncate the infinite domain into a finite domain, such as one-way approximation (cf. [6, 7, 10]), PML (cf. [4, 5, 15, 26, 30, 31, 33]), and so on. In this paper, PML is used to truncate the domain and absorb the outgoing waves. The technique of PML was proposed by Bérenger in 1994 (see, [4]). PML has the astonishing property of generating almost no reflection in theory at the interface between the interior medium (the interested domain) and the artificial absorbing medium. The key idea of the PML technique is to introduce an artificial layer with an attenuation parameter around the interior area. The magnitude of the wave is attenuated in the layer while the phase of the wave is conserved. After adding PML to the interior domain, we can impose boundary conditions, like Dirichlet boundary condition, Robin boundary condition and so on, on the outer boundary. Then we obtain a bounded boundary problem, which is usually inverted but ill-conditioned. We refer the interested readers to the

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paper [30] for the solvability and the uniqueness for the Helmholtz equation with PML.

For many years, finite difference methods (cf. [3, 11, 14, 16, 19, 24, 25, 26, 27, 32]) and finite element methods (cf. [1, 2, 8, 12, 17]) have been widely used to discrete the Helmholtz equation (1.1). As is known to all, the solution of the Helmholtz equation oscillates severely for large wavenumbers, and the quality of the numerical results usually deteriorates as the wavenumber k increasing (cf. [1, 2, 8, 12, 17]). Hence, there is a growing interest in discretization methods where the computational complexity increases only moderately with increasing wavenumber (cf. [1, 12, 13, 19, 25]).

Finite-difference frequency-domain modeling for the generation of synthetic seismograms and crosshole tomography has been an active field of research since the 1980s (see, [25]). Finite difference methods are easily implemented and its computational complexity is much less than that of finite element methods, although the finite difference method's accuracy is usually lower than that of the finite element method. In addition, by optimizing the parameters in the finite difference formulas, we can easily minimize the numerical dispersion (see, [19, 25]). For accurate modeling, the conventional 5-point finite difference scheme requires 10 gridpoints per wavelength. Therefore, for the Helmholtz equation with large wavenumbers, the resulting matrix is very large and ill-conditioned. Usually, direct methods do not perform well, and iterative methods with preconditioners are alterative (cf. [9, 11, 32]). In 1996, Jo, Shin and Suh proposed the rotated 9-point finite difference scheme for the Helmholtz equation (see, [19]). The approach consists of linearly combining the two discretizations of the second derivative operator on the classical Cartesian coordinate system and the 45^o rotated system. They also gave a group of optimal parameters based on the normalized phase velocity. This optimal 9-point scheme reduces the number of gridpoints per wavelength to 5 while preserving the accuracy of the conventional 5-point scheme with 10 gridpoints per wavelength. Therefore, computer memory and CPU time are saved. In 1998, Shin and Sohn extended the idea of the rotated 9-point scheme to the 25-point formula, and they obtained a group of optimal parameters by the singular-value decomposition method (see, [25]). Furthermore, the 25-point formula reduces the number of gridpoints per wavelength to 2. However, the resulting matrix's bandwidth is much wider than that of the 9-point scheme, and there are some difficulties when considering the absorbing conditions. To reduce the numerical error, higher-order finite difference schemes (cf. [3, 26]) were also constructed and widely used. However, to obtain higher-order accuracy, the higher-order schemes require the source term to be smooth enough. Many the practical problems (cf. [22, 23]) are not the case. On the other hand, though that the rotated difference scheme is a popular solver for the Helmholtz equation, it is not a good choice for the Helmholtz equation with PML. We shall illustrate this in detail in this paper.

This paper is organized as follows. In Section 2, we investigate the rotated 9-point finite difference scheme and show that it is not pointwise consistent with the Helmholtz equation with PML. In Section 3, we present a 9-point difference scheme by using the approach suggested in [26], and prove that it is consistent with the Helmholtz equation with PML and is a second order scheme. For this 9-point difference scheme, we then analyze the error between the numerical wavenumber and the exact wavenumber, and propose global and refined choice strategies for choosing optimal parameters of the scheme based on minimizing the numerical dispersion. In Section 4, numerical experiments are given to demonstrate the efficiency of the